Graduate Preliminary Examination

Algebra I

22.9.2004; 3 hours

Problem 1. Let p be a prime number. If G is an infinite p-group such that every proper non-trivial subgroup of G has order p, prove that:

(a) $p > 2$;

(b) G must be a simple group.

(Recall that a group G is called simple if it has no non-trivial normal subgroup.)

Problem 2. Let $\mathbb{Q}$ be the additive group of rationals.

(a) Prove that every finitely generated subgroup of $\mathbb{Q}$ is cyclic.

(b) Prove that if G is a group with center $Z(G)$ such that $G/Z(G)$ is isomorphic to a subgroup of $\mathbb{Q}$, then G is abelian.

(c) Prove that no non-trivial subgroup of $\mathbb{Q}$ can be isomorphic to the full group of automorphisms of a group.

Problem 3. Let R be a quadratic integer $\mathbb{Z}[\sqrt{-5}]$. Let $I = (2, 1 + \sqrt{-5})$ be an ideal of R.

(a) Is I a principal ideal in $\mathbb{Z}[\sqrt{-5}\,]$? Justify your answer.

(b) Is $I^2 = II$ a principal ideal in $\mathbb{Z}[\sqrt{-5}\,]$ Justify your answer.

Problem 4. Let R be a commutative ring with identity. If I is an ideal of R , then $\sqrt{I} = \{r \in R \ : \ r^n \in I \quad \text{for some positive integer } n\}$. A proper ideal I is called a primary ideal if whenever $ab \in I$ we have either $a \in I$ or $b \in \sqrt{I}$.

(a) Prove that if I is a primary ideal, then $\sqrt{I}$ is a prime ideal.

(b) Is $(4, x)$ a prime ideal in $\mathbb{Z}[x]$? Explain.

(c) Is $(4, x)$ a primary ideal in $\mathbb{Z}[x]$? Explain.