PRELIMINARY EXAMINATION
ALGEBRA I
FalI 2005
September 14^{th}, 2005

Duration: 3 hours

1. Determine all groups with exactly three distinct subgroups.

2. Let A be an abelian group denoted additively. Let ϕ be an endomorphism of A. Show that if ϕ is nilpotent, then $1 + \phi$ is an automorphism of A.

Hint: Consider the factorization of $1 + \phi^n$ (with n odd) in the ring End A. Note that 1 means the identity map of A.

3. A ring R is called <u>radical</u> if for every $x \in R$, there exists $y \in R$ such that $x + y + xy = 0$.

a) Let R be a ring. If every element of R is nilpotent, then show that R is radical.

b) Show that $R = \left\{ \dfrac{2x}{2y+1} | x, y \in \mathbb{Z} \text{ such that } (2x, 2y+1) = 1 \right\}$ is a radical ring.

c) Prove or disprove: In a radical ring every element is nilpotent.

4. Let R be a commutative ring with identity 1. A subset S of R is called a <u>multiplicative set</u> if it is closed under multiplication, contains 1, and does not contain the zero element.

a) Prove that an ideal I of R is prime if and only if there is a multiplicative set S such that I is maximal among ideals disjoint from S.

b) Prove that the set of all nilpotent elements of R equals the intersection of all the prime ideals of R.

Hint: If s is not nilpotent, then $\{1, s, s^2, \cdots\}$ is a multiplicative set.