1. Determine all groups with exactly three distinct subgroups.

2. Let $A$ be an abelian group denoted additively. Let $\phi$ be an endomorphism of $A$. Show that if $\phi$ is nilpotent, then $1 + \phi$ is an automorphism of $A$.

   **Hint:** Consider the factorization of $1 + \phi^n$ (with $n$ odd) in the ring $\text{End} A$. Note that $1$ means the identity map of $A$.

3. A ring $R$ is called **radical** if for every $x \in R$, there exists $y \in R$ such that $x + y + xy = 0$.

   a) Let $R$ be a ring. If every element of $R$ is nilpotent, then show that $R$ is radical.

   b) Show that $R = \left\{ \frac{2x}{2y + 1} \bigg| x, y \in \mathbb{Z} \text{ such that } (2x, 2y + 1) = 1 \right\}$ is a radical ring.

   c) Prove or disprove: In a radical ring every element is nilpotent.

4. Let $R$ be a commutative ring with identity $1$. A subset $S$ of $R$ is called a multiplicative set if it is closed under multiplication, contains $1$, and does not contain the zero element.

   a) Prove that an ideal $I$ of $R$ is prime if and only if there is a multiplicative set $S$ such that $I$ is maximal among ideals disjoint from $S$.

   b) Prove that the set of all nilpotent elements of $R$ equals the intersection of all the prime ideals of $R$.

   **Hint:** If $s$ is not nilpotent, then $\{1, s, s^2, \cdots\}$ is a multiplicative set.