TMS

Fall 2010

Algebra I

1. Let $G$ be a finite group and suppose $H \leq G$ satisfies the condition that $C_G(x) \leq H$ for all $x \in H - \{1\}$. Show that $gcd(|H|, |G : H|) = 1$.

   Hint: Choose $P \in Syl_p(H)$ and show that $P \in Syl_p(G)$

2. In this question, $G$ stands for a finite group where $C_G(a)$ is abelian for every $a \in G \setminus \{1\}$.

   (a) Give two examples of such a group; one non-abelian nilpotent, one non-nilpotent solvable.

   (b) Assume $Z(G) = 1$. Show that commuting is an equivalence relation on $G \setminus \{1\}$. What are the equivalence classes?

   (c) Let $Z(G) = 1$ and $A$ be a maximal abelian subgroup of $G$. Show that $A = C_G(a)$ for every $a \in A$ and $\gcd(|A|, [G : A]) = 1$.

3. Let $R$ be a commutative ring.

   Prove that the following are equivalent for $R$.

   **(1)** There is a proper ideal $P$ in $R$ such that $P \supseteq I$, for every proper ideal $I$ of $R$.

   **(2)** The set of nonunits of $R$ forms an ideal.

   **(3)** There exists a maximal ideal $M$ of $R$ such that $1 + x$ is a unit, for all $x \in M$.

4. Let $R$ be ring with unity and define

   $$N(R) = \{a \in R : a^n = 0 \quad \text{for some} \quad n \geq 1\}$$

   $J(R) = \cap M$, intersection of all maximal ideals in $R$.

   (a) Show that if $R$ is commutative, then $N(R)$ is an ideal and that

   $$N(R) = \cap P, \quad \text{intersection of all prime ideals in R.}$$

   (b) Give an eample to show that if $R$ is not commutative, then $N(R)$ need not be an ideal.

   (c) Give examples $R, S$ of commutative rings such that

   $$0 \neq N(R) = J(R)$$

   $$0 \neq N(S) \neq J(S)$$