METU MATHEMATICS DEPARTMENT
ALGEBRA I
SEPTEMBER 2012 - TMS EXAM

1. Prove that every group of order $3^2 \cdot 5 \cdot 17$ is abelian

2. a) Prove that Aut $S_3 \cong S_3$.

b) Prove that every nonabelian simple group G is isomorphic to a subgroup of Aut$G$.

3. Let R be an integral domain with 1. A non-zero, non-unit element $s \in R$ is said to be **special** if, for every element $a \in R$, there exist $r, q \in R$ with $a = qs + r$ and such that r is either 0 or a unit of R.

Prove the following:

a) If $s \in R$ is a special element, then the principal ideal (s) generated by s is maximal in R.

b) Every polynomial in $\mathbb{Q}[x]$ of degree 1 is special in $\mathbb{Q}[x]$.

c) There are no special elements in $\mathbb{Z}[x]$. (Hint: Apply the definition with $a = 2$ and $a = x$)

4. Let $A_1, \cdots, A_n$ be ideals of the commutative ring R, and let $D = \bigcap_{i=1}^{n} A_i$. Recall that the radical $\sqrt{I}$ of an ideal I of R is defined as $\sqrt{I} = \{x \in R \,|\, x^k \in I \text{ for some positive integer } k\}$.

(a) Prove that $I \subseteq \sqrt{I}$, $\sqrt{I} = \sqrt{\sqrt{I}}$ and $\sqrt{I} \subseteq \sqrt{J}$ whenever $I \subseteq J$ for ideals I and J in R.

(b) Prove that $\sqrt{D} = \bigcap_{i=1}^{n} \sqrt{A_i}$.

(c) Suppose that D is a primary ideal and D is not the intersection of elements in any proper subset of $\{A_1, \ldots, A_n\}$. Show that $\sqrt{A_i} = \sqrt{D}$ for each $i = 1, \ldots, n$.

(Recall that an ideal $I(\neq R)$ of R is primary if for any $x, y \in R$, $xy \in I$ and $x \notin I \Rightarrow y^\ell \in I$ for some positive integer ℓ)