METU MATHEMATICS DEPARTMENT
GRADUATE PRELIMINARY EXAMINATION
ALGEBRA I, SEPTEMBER 2013

SEPTEMBER 19, 2013

1.a) Let $G = H_1 \cup H_2 \cup H_3$ be a finite group, where each H_i is a proper subgroup of G. Show that $H_i \neq H_j$ if $i \neq j$.

1.b) Show that each H_i has index two in G.

2.a) Show that any group of order $175 = 5^2\,7$ has a unique Sylow 5 and a unique Sylow 7 subgroup. Use this to show that any group of order $175 = 5^2\,7$ is abelian. List all groups of order 175 up to isomorphism.

2.b) Show that the automorphism group of $\mathbb{Z}_2 \times \mathbb{Z}_2$ is isomorphic to the symmetric group on three letters S_3.

2.c) Let G be a finite group and $N \triangleleft G$ a normal subgroup isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_2$ such that $\gcd(6, |G|/4) = 1$. Use Part (b) to show that $N \subseteq Z(G)$; in other words, N is contained in the center of G.

3.a) Let $\mathbb{F}$ be a field with characteristic different from seven. Show that the polynomial $f(x, y) = 7x - x^2y + 2xy^3 - 5x^3y^4 + y^{100} \in \mathbb{F}[x, y]$ is irreducible. Is the same polynomial irreducible as an element of $\mathbb{Z}[x, y]$? Explain.

3.b) State the Eisenstein Criterion for UFD's and use it to show that the polynomial

$$g(x, y, z) = z^{1000} + x + \sum_{i=1}^{999} x^i y^i z^i$$

is irreducible in $\mathbb{C}[x, y, z]$.

4) Let p be a prime integer and $S = \mathbb{Z} - (p)$; the set of all integers not divisible by p. Show that S is a multiplicative subset of $\mathbb{Z}$ and consider the localization

$$\mathbb{Z}_{(p)} \doteq S^{-1}\mathbb{Z} = \{\frac{a}{b} \mid a \in \mathbb{Z},\ b \in S\}$$

as a subring of the field of rational numbers. Show that $\mathbb{Z}_{(p)}$ has a unique maximal ideal $m \subset \mathbb{Z}_{(p)}$, consisting of all non units in $\mathbb{Z}_{(p)}$. What is the quotient field $\mathbb{Z}_{(p)}/m$? Prove your answer.