1.a) Let $G = H_1 \cup H_2 \cup H_3$ be a finite group, where each $H_i$ is a proper subgroup of $G$. Show that $H_i \neq H_j$ if $i \neq j$.

1.b) Show that each $H_i$ has index two in $G$.

2.a) Show that any group of order $175 = 5^2 \cdot 7$ has a unique Sylow 5 and a unique Sylow 7 subgroup. Use this to show that any group of order $175 = 5^2 \cdot 7$ is abelian. List all groups of order 175 up to isomorphism.

2.b) Show that the automorphism group of $\mathbb{Z}_2 \times \mathbb{Z}_2$ is isomorphic to the symmetric group on three letters $S_3$.

2.c) Let $G$ be a finite group and $N \triangleleft G$ a normal subgroup isomorphic to $\mathbb{Z}_3 \times \mathbb{Z}_2$ such that $\gcd(6, |G|/4) = 1$. Use Part (b) to show that $N \subseteq Z(G)$; in other words, $N$ is contained in the center of $G$.

3.a) Let $F$ be a field with characteristic different from seven. Show that the polynomial $f(x, y) = 7x - x^2y + 2x^3 - 5x^2y^4 + y^{100} \in F[x, y]$ is irreducible. Is the same polynomial irreducible as an element of $F[x, y]$? Explain.

3.b) State the Eisenstein Criterion for UFD's and use it to show that the polynomial

$$g(x, y, z) = x^{100} + x + \sum_{i=1}^{999} x^iy^iz^4$$

is irreducible in $C[x, y, z]$.

4) Let $p$ be a prime integer and $S = \mathbb{Z} \setminus \{p\}$; the set of all integers not divisible by $p$. Show that $S$ is a multiplicative subset of $\mathbb{Z}$ and consider the localization

$$\mathbb{Z}(p) = S^{-1}\mathbb{Z} = \{\frac{a}{b} \mid a \in \mathbb{Z}, \ b \in S\}$$

as a subring of the field of rational numbers. Show that $\mathbb{Z}(p)$ has a unique maximal ideal $m \subset \mathbb{Z}(p)$, consisting of all non units in $\mathbb{Z}(p)$. What is the quotient field $\mathbb{Z}(p)/m$? Prove your answer.