1. a) Show that the alternating group $A_n$ has no proper subgroup of index less than $n$, provided that $n \geq 5$. (Hint: Assume that such a subgroup $H$ exists and then consider the action of $A_n$ on the left cosets of $H$ in $A_n$.)

b) Use Part (a) to prove that $S_n$ has no proper subgroup of index less than $n$ other than $A_n$, provided that $n \geq 5$.

2. a) Let $p, q$ be primes with $p \geq q^2$ and $G$ be a group of order $pq^2$. Prove that $G$ has a normal Sylow $p$-subgroup.

b) In addition to the assumptions in Part (a) assume further that the greatest common divisor $(q^2, p - 1) = 1$. Show that the group in Part (a) is abelian.

3. a) Show that the map

$$f : \mathbb{C} \to M_2(\mathbb{R}), \quad f(a + ib) = \begin{bmatrix} a & -b \\ b & a \end{bmatrix},$$

is an injective ring homomorphism, where $M_2(\mathbb{R})$ is the ring of $2 \times 2$-real matrices.

b) Is the image an ideal in $M_2(\mathbb{R})$? Why?

4. Let $R$ be the ring of continuous functions on the interval $[0, 1]$.

a) What are the units of the ring $R$?

b) For any $a \in [0, 1]$, let $I_a$ be the set of elements $f \in R$ with $f(a) = 0$. Show that $I_a$ is a maximal ideal in $R$.

c) Show that the set of elements $f \in R$ with $f(1/2) = 0 = f(1/4)$ is an ideal. Is it prime?

d) Show that any maximal ideal of $R$ is of the form $I_a$, for some $a \in [0, 1]$. 