M E T U Department of Mathematics

Graduate Preliminary Exam, Algebra I	Fall 2019	18.09.2019	10:00
Last Name : Name : Student No :	Signature :		
5 QUESTIONS			TOTAL 100 POINTS
1 | 2 | 3 | 4 | 5	**Duration 3 hours**		

(1) (20 pts) Let G be a finite group and let p be the smallest prime dividing the order of G. If H is a subgroup of G of index p, show that H is a normal subgroup of G. (Hint: Consider the action of G on the set of left cosets of H).

(2) (20 pts) Let G be a finite group such that for all $n \geq 1$, G has at most one subgroup of order n. Show that G is cyclic.

(3) Recall the following: If G is a group, a subgroup $M \lneq G$ is a *maximal* subgroup if for any subgroup $N \leq G$, $M \leq N \leq G$ implies $N = M$ or $N = G$. (i.e., M is a maximal element in the poset of proper subgroups of G).

(a) (10 pts) Using Zorn's Lemma, prove the following:
If G is a finitely generated group then G has a maximal subgroup.

(b) (10 pts) Show that the group $(\mathbb{Q}, +)$ has no maximal subgroups.

(4) Let R be an integral domain. An element $s \in R$ is called *special* if

- s is non-zero and non-unit and
- for any $a \in R$ there exists $q, r \in R$ such that $a = qs + r$ where $r = 0$ or r is a unit.

(a) (7 pts) Show that if $s \in R$ is special then (s) is a maximal ideal of R.
(b) (7 pts) Show that every polynomial in $\mathbb{Q}[x]$ of degree 1 is special.
(c) (6 pts) Show that $\mathbb{Z}[x]$ has no special elements.

(5) Let F be a field and let $R = \{a_n x^n + \ldots + a_1 x + a_0 \in F[x] \mid a_1 = 0\}$. Clearly R is a subring of $F[x]$.
(a) (10 pts) Is R a unique factorization domain? Explain.
(b) (10 pts) Prove or disprove: If S is a Euclidean domain, then every subring of S is also a Euclidean domain.