Graduate Preliminary Examination
Algebra 1, November 2020

1. Prove that if G is a group of order 385 then the center of G contains a Sylow 7-subgroup of G and a Sylow 11-subgoup is normal in G.

2. Let G be a group and H be a subgroup of G. Let $C_G(H)$ be the centralizer of H and let $N_G(H)$ be the normalizer of H.

(a) For each $a \in N_G(H)$ define a function by $f_a : H \to G$ by $f_a(h) := aha^{-1}$. Show that f_a is in $Aut(H)$.

(b) Show that the map $f_a : N_G(H) \to Aut(H)$ is a group homomorphism.

(c) Show that the factor group $N_G(H)/C_G(H)$ is isomorphic to a subgroup of $Aut(H)$.

3. Let R be the ring $\mathbb{Z}[\sqrt{-5}]$ and let $\mathfrak{p}$ be the ideal $(3, 1 + \sqrt{-5})$ in R.

(a) Show that $\mathfrak{p}$ is a prime ideal.

(b) Show that $\mathfrak{p}$ is not a principal ideal. Is $\mathfrak{p}^2$ principal?

4. Let R be a commutative ring with unity. A non-zero ideal I of R is said to be a *minimal ideal* if there does not exist an ideal J of R such that $\{0\} \subsetneq J \subsetneq I$.

(a) Suppose that K is a minimal ideal of R such that

$$K^2 = \{kl \mid k, l \in K\} \neq \{0\}$$

Show that there exist non-zero $e, k \in K$ such that $ek = k$.

(b) Let e, k, K be as in part (a). First show that the set

$$Ann_K(k) = \{x \in K : xk = 0\}$$

is an ideal of R contained in K. Then deduce that e is an idempotent element, that is, $e^2 = e$.

(c) Suppose that R is an integral domain with a minimal ideal. Show that R is a field.