Graduate Preliminary Examination  
Algebra 1, November 2020

1. Prove that if $G$ is a group of order 385 then the center of $G$ contains a Sylow 7-subgroup of $G$ and a Sylow 11-subgroup is normal in $G$.

2. Let $G$ be a group and $H$ be a subgroup of $G$. Let $C_G(H)$ be the centralizer of $H$ and let $N_G(H)$ be the normalizer of $H$.
   (a) For each $a \in N_G(H)$ define a function by $f_a : H \to G$ by $f_a(h) := aha^{-1}$. Show that $f_a$ is in $Aut(H)$.
   (b) Show that the map $f_a : N_G(H) \to Aut(H)$ is a group homomorphism.
   (c) Show that the factor group $N_G(H)/C_G(H)$ is isomorphic to a subgroup of $Aut(H)$.

3. Let $R$ be the ring $\mathbb{Z}[\sqrt{-5}]$ and let $p$ be the ideal $(3, 1 + \sqrt{-5})$ in $R$.
   (a) Show that $p$ is a prime ideal.
   (b) Show that $p$ is not a principal ideal. Is $p^2$ principal?

4. Let $R$ be a commutative ring with unity. A non-zero ideal $I$ of $R$ is said to be a minimal ideal if there does not exist an ideal $J$ of $R$ such that $\{0\} \subsetneq J \subsetneq I$.
   (a) Suppose that $K$ is a minimal ideal of $R$ such that
   \[
   K^2 = \{kl \mid k, l \in K\} \neq \{0\}
   \]
   Show that there exist non-zero $e, k \in K$ such that $ek = k$.
   (b) Let $e, k, K$ be as in part (a). First show that the set
   \[
   Ann_K(k) = \{x \in K : xk = 0\}
   \]
   is an ideal of $R$ contained in $K$. Then deduce that $e$ is an idempotent element, that is, $e^2 = e$.
   (c) Suppose that $R$ is an integral domain with a minimal ideal. Show that $R$ is a field.