

METU Mathematics Department
Graduate Preliminary Examination
Algebra I, September 2025

1. Construct a group of order 21 that is not abelian. Sketch its subgroup diagram.
2. Let D_{2n} be the dihedral group of order $2n$.
 - (a) Determine the center of D_{2n} .
 - (b) Prove that D_{2n} is nilpotent if and only if n is a power of 2.
3. Recall that the quaternion group $Q_8 = \{\pm 1, \pm i, \pm j, \pm k\}$ is a nonabelian group that is **not** isomorphic to D_8 . What is the smallest n such that Q_8 is isomorphic to a subgroup of S_n ?
4. Consider the ring $R = \{a + b\sqrt{-3} \mid a, b \in \mathbf{Z}\}$.
 - (a) Is the ideal $I = (2, 1 + \sqrt{-3})$ principal? prime? maximal?
 - (b) Set $J = (2)$. Show that $I \neq J$ but $I^2 = IJ$.
5. Let $w = \sqrt{-2}$ and let $R = \mathbf{Z}[w] = \{a + bw \mid a, b \in \mathbf{Z}\}$.
 - (a) Show that R is a Euclidean domain.
 - (b) Find a generator for the ideal $I = (1 + 2w, 2 + w)$. Determine the number of elements in the quotient ring R/I .