Graduate Preliminary Examination

Algebra I

18.2.2004
3 hours

Problem 1. (a) Let G be a finite nilpotent group. Show that if m divides the order of G, then G has a subgroup of order m.

(b) Give an example of a finite group G such that m divides the order of G but G does not have a subgroup of order m.

Problem 2. Let Σ be the set of Sylow p-subgroups of some finite group, $|\Sigma| \geq 2$ and let $P \in \Sigma$. Clearly P acts on Σ by conjugation.

(a) Find the fix points of P in the set $\Sigma \setminus \{P\}$ if there are any.

(b) Find the length of the orbits of P containing an element of $\Sigma \setminus \{P\}$.

Problem 3. Here $\mathbb{Q}$ is the ring of rational numbers. Let p be the polynomial $X^3 + 9X + 6$ over $\mathbb{Q}$, and let θ be a root of p.

(a) Write θ^3, θ^4 and θ^5 as $\mathbb{Q}$-linear combinations of 1, θ and θ^2.

(b) Is $1 + \theta$ invertible in $\mathbb{Q}[X]/(p)$? If it is, find the inverse; if it is not, explain why.

Problem 4. Let R be a countable integral domain. Prove that R is a principal ideal domain, provided that the following two conditions hold:

- Any two non-zero elements a and b of R have a greatest common divisor, which can be written in the form $ra + sb$ for some r and s in R.
- If $a_1, a_2, \ldots$ are nonzero elements of R such that $a_{n+1} \mid a_n$ for all positive integers n, then there is a positive integer N, such that if $n \geqslant N$, then a_n is a unit times a_N.