Graduate Preliminary Examination

Algebra I

16.2.2005: 3 hours

Problem 1. A group G is called a residually finite group if for any $1 \neq x \in G$ there exists a normal subgroup N such that $x \notin N$ and $|G/N|$ is finite.

Prove that every finitely generated abelian group is residually finite.

Problem 2. Let $GL(n, F_p)$ denote the group of all non-singular $n \times n$ matrices over a field F_p with p elements.

(a) Show that $n \times n$ strictly upper triangular matrices with 1 on the diagonal (unitriangular matrices) in $GL(n, F_p)$ is a Sylow p-subgroup of $GL(n, F_p)$.

(b) Prove that the number of Sylow subgroups in $GL(2, F_p)$ is $p+1$. Exhibit two distinct Sylow p-subgroups of $GL(2, F_p)$.

Problem 3. Let $R = \mathbb{Q}[x]/(x^2 - 1)$.

(a) Find e and f in R, both non-zero such that

$$e^2 = e,\ \ f^2 = f, ef = 0,\ \ e + f = 1.$$

(b) Show that $\phi_e : R \to R$ where $\phi_e(r) = re$ is a ring homomorphism.

(c) Find the kernel of the homomorphism ϕ_e

Problem 4. (a) Let S and K be two rings with identity. Let $\phi : S \to K$ be a bijection satisfying $\phi(ab)\phi(1) = \phi(a)\phi(b)$. Show that $\phi(1)$ is an invertible element of K. (Two sided)

(b) Prove or give counter example to the following statement:

If R and S are two rings with identities 1_R and 1_S respectively, then $\phi(1_R) = 1_S$ for every ring homomorphism ϕ from R into S.