GRADUATE PRELIMINARY EXAMINATION

ALGEBRA I
Spring 2010

1. Prove that a finite group G is nilpotent if and only if $ab = ba$ whenever a and b are elements of G with $(|a|, |b|) = 1$.

2. Let p and q be distinct primes . Show that there is no simple group of order p^3q.
(Hint: The case $p^3q = 24$ can be treated separately)

3. Let R be a commutative ring with identity 1.

(a) Let $P_1 \subseteq P_2 \subseteq \cdots$ and $Q_1 \subseteq Q_2 \subseteq \cdots$ be chains of prime ideals in R. Show that $\bigcup P_i$ and $\bigcap Q_i$ are also prime ideals of R.

(b) Assume that P and Q are prime ideals in R such that $P \subsetneq Q$. Show that there exist prime ideals P^* and Q^* in R such that $P \subseteq P^* \subsetneq Q^* \subseteq Q$ and there is no prime ideal properly lying between P^* and Q^*. (Hint: You may need to use Zorn's lemma.)

4. Let K be a field. A discrete valuation on K is a function $\nu : K^* \longrightarrow \mathbb{Z}$ satisfying

(i) $\nu(ab) = \nu(a) + \nu(b)$

(ii) $\nu(x+y) \geq min\{\nu(x), \nu(y)\}$ for all x, y in K^* with $x + y \neq 0$.

The set $R = \{x \in K^* \mid \nu(x) \geq 0\} \cup \{0\}$ is called a valuation ring of ν.

(a) Prove that R is a subring of K which contains the identity.

(b) Prove that for each non-zero element $x \in K$ either x or x^{-1} is in R.

(c) Prove that an element x is a unit of R if and only if $\nu(x) = 0$.

(d) Give an example of a discrete valuation on the field of rational numbers.