1. Prove that a finite group $G$ is nilpotent if and only if $ab = ba$ whenever $a$ and $b$ are elements of $G$ with $(|a|, |b|) = 1$.

2. Let $p$ and $q$ be distinct primes. Show that there is no simple group of order $p^5q$. 
   (Hint: The case $p^5q = 24$ can be treated separately)

3. Let $R$ be a commutative ring with identity 1.
   
   (a) Let $P_1 \subseteq P_2 \subseteq \cdots$ and $Q_1 \subseteq Q_2 \subseteq \cdots$ be chains of prime ideals in $R$. Show that $\bigcup P_i$ and $\bigcap Q_i$ are also prime ideals of $R$.

   (b) Assume that $P$ and $Q$ are prime ideals in $R$ such that $P \subseteq Q$. Show that there exist prime ideals $P^*$ and $Q^*$ in $R$ such that $P \subseteq P^* \subseteq Q^* \subseteq Q$ and there is no prime ideal properly lying between $P^*$ and $Q^*$. (Hint: You may need to use Zorn’s lemma.)

4. Let $K$ be a field. A discrete valuation on $K$ is a function $\nu : K^* \rightarrow \mathbb{Z}$ satisfying
   
   (i) $\nu(ab) = \nu(a) + \nu(b)$
   
   (ii) $\nu(x + y) \geq \min\{\nu(x), \nu(y)\}$ for all $x, y$ in $K^*$ with $x + y \neq 0$.

The set $R = \{x \in K^* \mid \nu(x) \geq 0\} \cup \{0\}$ is called a valuation ring of $\nu$.

   (a) Prove that $R$ is a subring of $K$ which contains the identity.

   (b) Prove that for each non-zero element $x \in K$ either $x$ or $x^{-1}$ is in $R$.

   (c) Prove that an element $x$ is a unit of $R$ if and only if $\nu(x) = 0$.

   (d) Give an example of a discrete valuation on the field of rational numbers.