M.E.T.U

Department of Mathematics

Preliminary Exam - Feb. 2011

ALGEBRA I

1. Let G be a finite group and p be a prime number dividing $|G|$, and let $X = \{x \in G \mid x^p = 1\}$. Prove that $|X|$ is divisible by p.

   (Hint: Let $P \in Syl_p G$. Consider the action of P on X by conjugation and verify that $X \cap C_G(P)$ is a subgroup of P .)

2. Let G be a finite group.

   a) Prove that $N_G(N_G(P)) = N_G(P)$ for each Sylow p-subgroup of G for a prime number p dividing $|G|$.

   b) Prove the implications (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii) $\Rightarrow$ (iv) for the following:

   (i) G is nilpotent.

   (ii) H is properly contained in $N_G(H)$ for each proper subgroup H of G.

   (iii) Every Sylow p-subgroup of G is normal in G for a prime number p dividing $|G|$.

   (iv) G is the direct product of its Sylow subgroups.

3. Let R be a commutative ring with 1, and let $\mathcal{S}$ be the set of all ideals of R in which every element is a zero divisor. Assume that $\mathcal{S} \neq \phi$.

   (a) Prove that $\mathcal{S}$ has maximal elements with respect to inclusion.

   (b) Prove that every maximal element of $\mathcal{S}$ with respect to inclusion is a prime ideal.

(c) Let D be the set of all zero divisors of R and P be a prime ideal of R. Prove that $P \cap D$ is a prime ideal of R and D is a union of prime ideals.

(d) Give an example of a commutative ring R with 1 such that D is not an ideal and describe the decomposition of D in your example as a union of prime ideals.

4. Let R be a commutative ring with 1, and let X be the set of prime ideals of A. For each subset S of R, let $V(S)$ denote the set of all prime ideals of R which contain S. Prove that

(a) If I is the ideal generated by S, then $V(S) = V(I)$.

(b) $V(0) = X$ and $V(1) = \phi$.

(c) If $(S_i)_{i \in I}$ is a family of subsets of R, then V $\left(\bigcup_{i \in I} S_i\right) = \bigcap_{i \in I} V(S_i)$.

(d) $V(P \cap Q) = V(PQ) = V(P) \cup V(Q)$ for any ideals P, Q of R.

(e) Suppose that R is Noetherian and let I be an ideal of R. Show that $V(I) = X$ if and only if $I^n = 0$ for some positive integer n.