1. Let $G$ be a finite group and $p$ be a prime number dividing $|G|$, and let $X = \{x \in G \mid x^p = 1\}$. Prove that $|X|$ is divisible by $p$.

(Hint: Let $P \in Syl_p G$. Consider the action of $P$ on $X$ by conjugation and verify that $X \cap C_G(P)$ is a subgroup of $P$.)

2. Let $G$ be a finite group.
   a) Prove that $N_G(N_G(P)) = N_G(P)$ for each Sylow $p$-subgroup of $G$ for a prime number $p$ dividing $|G|$.
   b) Prove the implications (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii) $\Rightarrow$ (iv) for the following:
      (i) $G$ is nilpotent.
      (ii) $H$ is properly contained in $N_G(H)$ for each proper subgroup $H$ of $G$.
      (iii) Every Sylow $p$-subgroup of $G$ is normal in $G$ for a prime number $p$ dividing $|G|$.
      (iv) $G$ is the direct product of its Sylow subgroups.

3. Let $R$ be a commutative ring with 1, and let $\mathcal{S}$ be the set of all ideals of $R$ in which every element is a zero divisor. Assume that $\mathcal{S} \neq \phi$.
   a) Prove that $\mathcal{S}$ has maximal elements with respect to inclusion.
   b) Prove that every maximal element of $\mathcal{S}$ with respect to inclusion is a prime ideal.
(c) Let $D$ be the set of all zero divisors of $R$ and $P$ be a prime ideal of $R$. Prove that $P \cap D$ is a prime ideal of $R$ and $D$ is a union of prime ideals.

(d) Give an example of a commutative ring $R$ with 1 such that $D$ is not an ideal and describe the decomposition of $D$ in your example as a union of prime ideals.

4. Let $R$ be a commutative ring with 1, and let $X$ be the set of prime ideals of $R$. For each subset $S$ of $R$, let $V(S)$ denote the set of all prime ideals of $R$ which contain $S$. Prove that

(a) If $I$ is the ideal generated by $S$, then $V(S) = V(I)$.

(b) $V(0) = X$ and $V(1) = \phi$.

(c) If $(S_i)_{i \in I}$ is a family of subsets of $R$, then $V \left( \bigcup_{i \in I} S_i \right) = \bigcap_{i \in I} V(S_i)$.

(d) $V(P \cap Q) = V(PQ) = V(P) \cup V(Q)$ for any ideals $P, Q$ of $R$.

(e) Suppose that $R$ is Noetherian and let $I$ be an ideal of $R$. Show that $V(I) = X$ if and only if $I^n = 0$ for some positive integer $n$. 

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