GRADUATE PRELIMINARY EXAMINATION
ALGEBRA I, FEBRUARY 2013

FEBRUARY 13, 2013

1.a. Let p, q, r be distinct primes. Show that any group G of order $|G| = p^2q^2r^2$ is abelian if and only if it is nilponent.

1.b. List all the nilpotent groups of order $p^2q^2r^2$, up to isomorphism.

1.c. Give an example of a nilpotent group of order $2^3 3^2 5^2$, which is not abelian.

2.a. Let G be a finite group of order 105 and let n_p denote the number of Sylow p-subgroups of G, where $p \in \{3, 5, 7\}$. Show that we cannot have simultaneously $n_p > 1$, for all p. Conclude that G is not simple.

2.b. Show that any group G of order 105 has indeed a unique Sylow-7 subgroup. (*Hint: If G does have not a normal subgroup of order* 7*, then show that it has a normal subgroup of order* 15*. In this case, a Sylow* 7*-subgroup acts on this subgroup of order* 15*, by conjugation. Next show that* G *is abelian, which yields a contradiction.*)

2.c. Is there any nonabelian group G of order 105? Explain your answer.

3.a. Let R be a commutative ring with unity 1. If $I \subseteq R$ is an ideal then its radical is defined to be subset

$$\sqrt{I} \doteq \{x \in R \mid x^n \in I, \text{ for some } n \in \mathbb{N}\}.$$

Show that $\sqrt{I}$ is an ideal of R.

3.b. Prove that for any prime ideal $P \subseteq R$ its radical is equal to itself: $P = \sqrt{P}$.

4.a. Let $f : R \to S$ be a surjective ring homomorphism, where R is a PID. Show that S is an integral domain if and only if S is a field.

4.b. Let F be any field. Show that any ring homomorphism $f : F[x] \to \mathbb{Z}$ is trivial (i.e., it is the zero homomorphism).

4.c. Construct infinitely many distinct ring homomorphisms from $\mathbb{Q}[x]$ to $\mathbb{Q}$.