1.a. Let $p, q, r$ be distinct primes. Show that any group $G$ of order $|G| = p^2 q^2 r^2$ is abelian if and only if it is nilponent.

1.b. List all the nilpotent groups of order $p^2 q^2 r^2$, up to isomorphism.

1.c. Give an example of a nilpotent group of order $2^3 3^2 5^2$, which is not abelian.

2.a. Let $G$ be a finite group of order 105 and let $n_p$ denote the number of Sylow $p$-subgroups of $G$, where $p \in \{3, 5, 7\}$. Show that we cannot have simultaneously $n_p > 1$, for all $p$. Conclude that $G$ is not simple.

2.b. Show that any group $G$ of order 105 has indeed a unique Sylow-7 subgroup. (Hint: If $G$ does have not a normal subgroup of order 7, then show that it has a normal subgroup of order 15. In this case, a Sylow 7-subgroup acts on this subgroup of order 15, by conjugation. Next show that $G$ is abelian, which yields a contradiction.)

2.c. Is there any nonabelian group $G$ of order 105? Explain your answer.

3.a. Let $R$ be a commutative ring with unity 1. If $I \subseteq R$ is an ideal then its radical is defined to be subset

$$\sqrt{I} = \{x \in R \mid x^n \in I, \text{for some } n \in \mathbb{N}\}.$$ 

Show that $\sqrt{I}$ is an ideal of $R$.

3.b. Prove that for any prime ideal $P \subseteq R$ its radical is equal to itself: $P = \sqrt{P}$.

4.a. Let $f : R \to S$ be a surjective ring homomorphism, where $R$ is a PID. Show that $S$ is an integral domain if and only if $S$ is a field.

4.b. Let $F$ be any field. Show that any ring homomorphism $f : F[x] \to \mathbb{Z}$ is trivial (i.e., it is the zero homomorphism).

4.c. Construct infinitely many distinct ring homomorphisms from $\mathbb{Q}[x]$ to $\mathbb{Q}$.