METU Mathematics Department
Graduate Preliminary Examination
Algebra I, January 2014

1. Prove or disprove the following claims:
   - No group can have exactly two subgroups of index two.
   - The abelian group $\mathbb{Q}/\mathbb{Z}$ is finitely generated.

2. Let H be a proper subgroup of a finite group G. Show that

$$G \neq \bigcup_{g \in G} gHg^{-1}.$$

3. Let R be a principal ideal domain and let $\mathfrak{p} \subset R$ be a non-zero prime ideal. Show that the localization $R_{\mathfrak{p}}$ is a principal ideal domain and has a unique irreducible element, up to associates.

4. Let $R = \mathbb{Z}[\sqrt{-5}]$ and let $\mathfrak{a} = (3, 1 + \sqrt{-5})$.
   - Show that $\mathfrak{a}$ is not a principal ideal.
   - Show that $\mathfrak{a}^2 = \mathfrak{a}\mathfrak{a}$ is a principal ideal and find a generator.