1.a) Let $G$ denote the free product of the group with two elements with itself: 
\[ G = \mathbb{Z}_2 * \mathbb{Z}_2 = \langle a, b \mid a^2, b^2 \rangle. \]
Show that the subgroup generated by the element $ab$, 
\[ N = \langle ab \rangle \triangleleft G \]
is an infinite cyclic group and it is normal with index two.

b) Let $H = \langle a \rangle \leq G$. Describe the action of $H$ on $N$ by conjugation. Show that $G = NH$ with $N \cap H = \{1\}$.

c) Conclude that $G$ is isomorphic to the semidirect product $N \rtimes H$ with the action described in part (b).

2.a) Show that any finite group of order 65 is cyclic.

b) Show that any finite group of order 130 has a unique subgroup of order 13.

c) Show that any finite group of order 130 has a subgroup of order 65.

d) Show that up to isomorphism there are only two groups of order 130. Describe them.

3.a) For any integer $n > 1$ consider the ring $\mathbb{Z}/n\mathbb{Z} \cong \mathbb{Z}_n$. Show that any ideal $I \subseteq \mathbb{Z}_n$ is principal. (Note that the ring $\mathbb{Z}_n$ is not necessarily a PID.)

b) Let $R$ be any ring so that there is an onto ring homomorphism $f : \mathbb{Z}_n \to R$. Use part (a) to show that $R$ is isomorphic to some $\mathbb{Z}_m$, where $m|n$.

c) Let $g : \mathbb{Z}_n \to \mathbb{Z}_m$ be an injective ring homomorphism and $g(1) = k \in \mathbb{Z}_m$ (we abuse the notation and let $k$ denote its residue class in $\mathbb{Z}_m$). Show that $k$ is an element of order $n$ in the additive group $\mathbb{Z}_m$, $m \mid nk$ and $m \mid k(k-1)$.

d) Let $m, n, k$ be integers with $m, n > 1$, $0 < k < m$ so that $m \mid nk$, $m \mid k(k-1)$ and $k \in \mathbb{Z}_m$ has order $n$. Show that the map $\phi : \mathbb{Z}_n \to \mathbb{Z}_m$ defined by $\phi(r) = kr$, $\forall r \in \mathbb{Z}_n$, is a well defined injective ring homomorphism.

e) List all subrings of $\mathbb{Z}_{30}$. Find a nontrivial ring homomorphism $\phi : \mathbb{Z}_{10} \to \mathbb{Z}_{30}$. Is there any other?

4.a) Let $N : \mathbb{Z}[\sqrt{5}] \to \mathbb{Z}$ be defined by the rule
\[ N(a + b\sqrt{5}) = (a + b\sqrt{5})(a - b\sqrt{5}) = a^2 - 5b^2. \]
Show that $N((a + b\sqrt{5})(c + d\sqrt{5})) = N(a + b\sqrt{5})N(c + d\sqrt{5})$, for all $(a + b\sqrt{5})$ and $(c + d\sqrt{5}) \in \mathbb{Z}[\sqrt{5}]$.

b) Use part (a) to show that if $|N(a + b\sqrt{5})| = p$ is a prime integer then $a + b\sqrt{5}$ is an irreducible element in $\mathbb{Z}[\sqrt{5}]$. 