METU MATHEMATICS DEPARTMENT
GRADUATE PRELIMINARY EXAMINATION
ALGEBRA I, FEBRUARY 2015

FEBRUARY 9, 2015

1.a) Let G denote the free product of the group with two elements with itself:

$$G = \mathbb{Z}_2 * \mathbb{Z}_2 =< a, b \mid a^2,\ b^2 > .$$

Show that the subgroup generated by the element ab,

$$N =< ab > \lhd G$$

is an infinite cyclic group and it is normal with index two.

b) Let $H =< a > \leqslant G$. Describe the action of H on N by conjugation. Show that $G = NH$ with $N \cap H = \{1\}$.

c) Conclude that G is isomorphic to the semidirect product $N \rtimes H$ with the action described in part (b).

2.a) Show that any finite group of order 65 is cyclic.

b) Show that any finite group of order 130 has a unique subgroup of order 13.

c) Show that any finite group of order 130 has a subgroup of order 65.

d) Show that up to isomorphism there are only two groups of order 130. Describe them.

3.a) For any integer $n > 1$ consider the ring $\mathbb{Z}/n\mathbb{Z} \simeq \mathbb{Z}_n$. Show that any ideal $I \subseteq \mathbb{Z}_n$ is principal. (Note that the ring $\mathbb{Z}_n$ is not necessarily a PID.)

b) Let R be any ring so that there is an **onto** ring homomorphism $f : \mathbb{Z}_n \to R$. Use part (a) to show that R is isomorphic to some $\mathbb{Z}_m$, where $m|n$.

c) Let $g : \mathbb{Z}_n \to \mathbb{Z}_m$ be an injective ring homomorphism and $g(1) = k \in \mathbb{Z}_m$ (we abuse the notation and let k denote its residue class in $\mathbb{Z}_m$). Show that k is an element of order n in the additive group $\mathbb{Z}_m$, $m \mid nk$ and $m \mid k(k-1)$.

d) Let m, n, k be integers with $m, n > 1$, $0 < k < m$ so that $m \mid nk$, $m \mid k(k-1)$ and $k \in \mathbb{Z}_m$ has order n. Show that the map $\phi : \mathbb{Z}_n \to \mathbb{Z}_m$ defined by $\phi(r) = kr$, $\forall r \in \mathbb{Z}_n$, is a well defined injective ring homomorphism.

e) List all subrings of $\mathbb{Z}_{30}$. Find a nontrivial ring homomorphism $\phi : \mathbb{Z}_{10} \to \mathbb{Z}_{30}$. Is there any other?

4.a) Let $N : \mathbb{Z}[\sqrt{5}] \to \mathbb{Z}$ be defined by the rule

$$N(a + b\sqrt{5}) = (a + b\sqrt{5})(a - b\sqrt{5}) = a^2 - 5b^2.$$

Show that $N((a + b\sqrt{5})(c + d\sqrt{5})) = N(a + b\sqrt{5})\ N(c + d\sqrt{5})$, for all $(a + b\sqrt{5})$ and $(c + d\sqrt{5}) \in \mathbb{Z}[\sqrt{5}]$.

b) Use part (a) to show that if $|N(a + b\sqrt{5})| = p$ is a prime integer then $a + b\sqrt{5}$ is an irreducible element in $\mathbb{Z}[\sqrt{5}]$.