METU Mathematics Department
Graduate Preliminary Examination
Algebra I, February 2018

1. Prove that a group of order 160 is not simple.

2. Let the group $G = S \times T$ be the direct product of subgroups S and T. Let H be a subgroup of G such that $SH = G = TH$.

   (a) Prove that $S \cap H$ and $T \cap H$ are normal subgroups of G.

   (b) If $S \cap H = 1 = T \cap H$ then prove that S and T are isomorphic.

   (c) If $S \cap H = 1 = T \cap H$ and H is normal in G, show that G is abelian.

3. Suppose that a is a non-zero non unit element of an integral domain R.

   (a) Show that the ideal (a, x) in $R[x]$ is not principal.

   (b) Use part (a) to show that if F is a field, then $F[x, y]$ is not a principal ideal domain.

4. (a) Let R be a commutative ring with unity and A be a proper ideal of R. Show that R/A is a commutative ring with unity.

   (b) In $\mathbb{Z} \oplus \mathbb{Z}$, let $I = \{(a, 0) \mid a \in \mathbb{Z}\}$. Show that I is a prime ideal but not a maximal ideal.

   (c) Prove that if R is a principal ideal domain then any nonzero prime ideal is maximal.