(1) Let \( \{ G_i \mid i \geq 1 \} \) be a family of finite groups and let \( G \) be the weak (restricted) direct product of this family.

(a) (15 Points) Show that every finitely generated subgroup of \( G \) is finite.

(b) (10 Points) Show that this may not be true for the direct product of such a family.

(2) (a) (5 Points) Let \( G \) be a group and \( H \subseteq C(G) \) be subgroup of \( G \). Show that if \( G/H \) is cyclic then \( G \) is abelian. (Here, \( C(G) \) denotes the center of the group \( G \)).

(b) (10 Points) Show that if \( \text{Aut}(G) \) is cyclic, then \( G \) is abelian. (Here, \( \text{Aut}(G) \) denotes the automorphism group of \( G \)).

(c) (10 Points) Give an example of an abelian group \( G \) such that \( \text{Aut}(G) \) is not abelian.

(3) (a) (10 Points) Let \( R \) be a principal ideal domain. Show that there is no infinite chain of ideals such that

\[
I_1 \subset I_2 \subset I_3 \subset \cdots
\]

(b) (15 pts) Let \( R \) be a unique factorization domain. Show that for any infinite chain of principal ideals of \( R \),

\[
(a_1) \subset (a_2) \subset (a_3) \cdots
\]

there is \( n \geq 1 \), such that \( (a_i) = (a_n) \) for \( i \geq n \).

(4) Let \( R = \{ f(x) \in \mathbb{Q}[x] \mid f(0) \in \mathbb{Z} \} \subseteq \mathbb{Q}[x] \).

(a) (5 Points) Show that \( R \) is an integral domain.

(b) (10 Points) Consider the following ideals of \( R \):

\[
I = \{ f(x) \in R \mid f(0) = 0 \}, \quad J = (x), \quad K = (2)
\]

Which of \( I, J, K \) are prime/maximal ideals? Explain.

(c) (10 Points) Is \( R \) a principal ideal domain? Explain.