Q1 (20 pts) Let $P$ be a Sylow $p$-subgroup of a finite group $G$, and let $Q$ be any $p$-subgroup of $G$. Then show that

$$Q \cap P = Q \cap N_G(P).$$

Q2 (20 pts) If $N$ is a normal subgroup of the finite group $G$ and $\gcd(|N|, [G : N]) = 1$, then prove that $N$ is the unique subgroup of $G$ of order $|N|$.

Q3 (20 pts) If $p$ and $q$ are primes then show that any group of order $p^2q$ is solvable.

Q4 (20 pts) A ring $R$ is called local if it has a unique maximal ideal.

(a) Prove that a commutative ring $R$ with unity is local if and only if the set of non-unit elements of $R$ is an ideal of $R$.

(b) Let $R$ be a commutative ring with identity $1_R$ and suppose that $M$ is a maximal ideal of $R$. Prove that if $1_R + M$ consists of units, then $R$ is a local ring.

Q5 (20 pts) Suppose $R$ is a commutative ring with identity. If $I$ is an ideal in $R[x]$ and $m$ is a nonnegative integer denote by $I(m)$ the set of all leading coefficients of polynomials of degree $m$ in $I$, together with 0.

(a) Show that $I(m)$ is an ideal in $R$.

(b) Show that $I(m) \subseteq I(m+1)$ for all $m$.

(c) If $J$ is an ideal with $I \subseteq J$ show that $I(m) \subseteq J(m)$ for all $m$. 