By signing below, I pledge that I will write this examination as my own work and without the assistance of others or the usage of unauthorized material or information. I understand that possession of any kind of electronic device during the exam is prohibited. I also understand that not obeying the rules of the examination will result in immediate cancellation and disciplinary procedures.

METU Department of Mathematics

Graduate Preliminary Exam		
Acad. Year : *2022-2023* Semester : *Fall* : Date : *March 2, 2023* Time : *10:00* Duration : *180 minutes*	**Full Name (USE CAPITAL LETTERS):** **Student ID:**	
5 QUESTIONS TOTAL 100 POINTS		**Signature:**

Q1 (20 *pts*) Let P be a Sylow p-subgroup of a finite group G, and let Q be any p-subgroup of G. Then show that

$$Q \cap P = Q \cap N_G(P).$$

Q2 (20 *pts*) If N is a normal subgroup of the finite group G and $\gcd(|N|, [G : N]) = 1$, then prove that N is the unique subgroup of G of order $|N|$.

Q3 (20 *pts*) If p and q are primes then show that any group of order p^2q is solvable.

Q4 (20 *pts*) A ring R is called **local** if it has a unique maximal ideal.

(a) Prove that a commutative ring R with unity is local if and only if the set of non-unit elements of R is an ideal of R.

(b) Let R be a commutative ring with identity 1_R and suppose that M is a maximal ideal of R. Prove that if $1_R + M$ consists of units, then R is a local ring.

Q5 (20 *pts*) Suppose R is a commutative ring with identity. If I is an ideal in $R[x]$ and m is a nonnegative integer denote by $I(m)$ the set of all leading coefficients of polynomials of degree m in I, together with 0.

(a) Show that $I(m)$ is an ideal in R.

(b) Show that $I(m) \subseteq I(m+1)$ for all m.

(c) If J is an ideal with $I \subseteq J$ show that $I(m) \subseteq J(m)$ for all m.