PRELIMINARY EXAMINATION
ALGEBRA II
FalI 2005
September 16^{th}, 2005

Duration: 3 hours

1. Let $f(x) = x^3 - 2x - 2 \in \mathbb{Q}[x]$. Let $K = \mathbb{Q}(\alpha)$ where α is a real root of f, and let F be the Galois closure of the extention $K/\mathbb{Q}$.

a) Determine the group of $\mathbb{Q}$-automorphisms of K.

b) Determine the Galois group $G(F/\mathbb{Q})$.

c) Determine the Galois group $G(F/K)$.

2. Let K be a field of characteristic p (where p is a prime number). Let $K^p = \{b^p | b \in K\}$.

a) Show that K^p is a subfield of K and K/K^p is an algebraic extension.

b) Let $a \in K, a \notin K^p$. Prove that $[K^p(a) : K^p] = p$.

3. Let R be a principal ideal domain, M a free R-module, and S a submodule of M. S is called a pure submodule if

whenever $ay \in S$ (with $a \in R \setminus \{0\}, y \in M$), then $y \in S$.

a) Show that $\{0\}$ and R are the only pure submodules of R, considered as an R-module)

b) Find a proper, nontrivial pure submodule of $R \oplus R$ (considered as an R-module).

c) Let N be a torsion-free R-module and $\varphi : M \to N$ be an R-module homomorphism. Prove that $Ker\varphi$ is a pure submodule of M.

4. Let R be a commutative ring with identity. Prove that every submodule of R is free iff $R = \{0\}$ or R is a principal ideal domain. (**Warning:** To prove that R is a PID, you have to show R is an integral domain first.)