TMS

Fall 2009

ABGEBRA II

1. Let R be a commutative ring with identity, and $A, B, C_1, \cdots, C_n$ be R-modules.

   a) Assume A is a submodule of B, and B satisfies the ACC on its submodules. Show that B/A satisfies the ACC on its submodules.

   b) Assume $C_1, \cdots, C_n$ satisfy the ACC on their submodules. Show that their direct sum also satisfies the ACC on its submodules.

   c) If the ring R satisfies the ACC on its ideals, then show that every finitely generated R-module satisfies the ACC on its submodules.

2. Let M be a left R-module.

   a) Prove that M is a simple module if and only if $M = Rm$ for all nonzero $m \in M$.

   b) Prove that M is simple if and only if $M \cong R/I$ for a maximal left ideal $I \subseteq R$.

   c) Prove that if M is simple, then $\mathrm{End}_R(M)$ is a division ring.

3. Let K be a subfield of a finite field L. Describe (as precisely as possible) the group of automorphisms of L when it is considered as:

   a) a field

   b) a vector space over K,

   c) an additive group

4. Let $f(x) = x^5 - 2 \in \mathbb{Q}[x]$.

   a) Find the order of the Galois group G_f of $f(x)$ over $\mathbb{Q}$.

   b) Show that G_f is isomorphic to the group H given by generators a of order 5 and b of order 4, with the relation $ba = a^2b$.