TMS

Fall 2010

Algebra II

1. Let R be a principal ideal domain, and let M be a finitely generated module over R. We know that, for some non-negative integers n and s, there are nonzero non-units $q_1, \ldots, q_n$ of R such that $q_k \mid q_{k+1}$ and M$\cong R/(q_1) \oplus \cdots \oplus R/(q_n) \oplus R^s$.

   (a) Letting K be the quotient field of R, find the dimension of $M \otimes_R K$ as a vector space over K.

   (b) Find the greatest integer t such that M has linearly independent elements $x_1, \ldots, x_t$: this means, if $a_1, \ldots, a_t \in R$ and $a_1 x_1 + \cdots + a_t x_t = 0$, then $a_1 = \cdots = a_t = 0$.

   Suppose further that R has only one prime ideal different from $\{0\}$, namely (p).

   (a) Give an example of such a ring R.

   (b) Show that $R/(p)$ is a field.

   (c) Show that, for some integers k_i such that $0 < k_1 < \cdots < k_m$, M$\cong R/(p^{k_1}) \oplus \cdots \oplus R/(p^{k_n}) \oplus R^s$.

   (d) Letting L be the field $R/(p)$, find the dimension of M/pM as a vector space over L.

   (e) Find the least integer t for which some subset $\{x_1, \ldots, x_t\}$ of M generates M over R.

   (f) Letting L be the field $R/(p)$, find the dimension of M/pM as a vector space over L.

   (g) Find the least integer t for which some subset $\{x_1, \ldots, x_t\}$ of M generates M over R.

2. Suppose E and F are finite extensions of a field K, and E and F are themselves subfields of some large field, so that the compositum EF is well defined: EF
E@-[ur]F@-[ul]
K@-[ul]@-[ur] Let us say that E is **free** from F over K if any elements of E that are linearly independent over K are still linearly independent (as elements of EF) over F.

   i. If E is free from F over K, show that F is free from E over K.

   ii. Prove that the following are equivalent:

      A. E is free from F over K,

      B. $[E : K] = [EF : F]$,

C. $[E:K][F:K]=[EF:K]$.

Suppose now also that E/K is Galois.

i. Prove that EF/F is Galois.

ii. Prove that E is free from F over K if and only if $E \cap F = K$.

3. Let p be a prime, $q = p^t$ for some $t \geq 1$, $F(q^k)$ denote the field with q^k elements, and $L(q) = \cup_{n \geq 1} F(q^{n!})$.

   (a) Show that $L(q)$ is a field. What is its prime subfield?

   (b) Show that $L(q)$ is an algebraic extension of $F(q)$.

   (c) Is $L(q)$ algebraically closed?

4. Let U be a right R-module and $X \subseteq U$ be any subset. Then show that

   (1) $ann_R(X) = \{r \in R | xr = 0 \quad \forall x \in X\}$ is a right ideal of R

   (2) If X is an R-submodule of U, then $ann_R(X)$ is an ideal of R.

   (3) If U is simple and $0 \neq x \in U$, then $ann_R(x)$ is a maximal right ideal of R and $U \cong R^{\bullet}/ann_R(x)$ where $R^{\bullet}$ denotes R as an R-module.