M.E.T.U

Department of Mathematics

Preliminary Exam - Sep. 2011

ALGEBRA II

Duration : 3 hr.

Each question is 25 pt.

1. Let n be a positive integer and F be a field of characteristic p with $p \not| n$.

   Let $f(x) = x^n - a$ for some $0 \neq a \in F$ and E be a splitting field for $f(x)$ over F.

   a) Show that $f(x)$ has no multiple roots (that is $f(x)$ has n distinct roots.)

   b) Show that E contains a primitive n-th root of unity ϵ.

   c) Assume that $\epsilon \in F$. Show that all irreducible factors of factors of $f(x)$ in $F[x]$ have the same degree and $[E : F]$ divides n.

2. Let α be an element of $\mathbb{C} - \overline{\mathbb{Q}}$ where $\overline{\mathbb{Q}}$ is the algebraic closure of $\mathbb{Q}$ in $\mathbb{C}$.

   a) Show that $\mathbb{Q}(\alpha)$ is the field of fractions of the integral domain $\mathbb{Q}[\alpha]$.

   (Hint : Use the homomorphism $\mathbb{Q}[x] \to \mathbb{C},\ x \mapsto \alpha$).

   b) Show that

   - each matrix $M = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in GL(2, \mathbb{Q})$ defines an automorphism

     $$\Phi_M : \mathbb{Q}(\alpha) \longrightarrow \mathbb{Q}(\alpha) \text{ given by } \alpha \mapsto \frac{a\alpha + b}{c\alpha + d}$$

   - and we obtain a group homomorphism
     $\Psi : GL(2, \mathbb{Q}) \longrightarrow \mathrm{Aut}(\mathbb{Q}(\alpha)),\ \Psi(M) = \Phi_M$.

   c) True or false? Explain.
   Ψ in (b) is an isomorphism.

3. Let M be a module over a commutative ring R satisfying the descending chain condition.

   Suppose that f is an endomorphism of M. Show that f is an isomorphism if and only if f is a monomorphism.

4. Let R be commutative ring with unity and M be an R-module. For $x \in M$ we define
$$\text{Ann}(x) = \{r \in R : rx = 0\}$$
   and we set $T(M) = \{x \in M : \text{Ann}(x) \neq 0\}$.

   a) Show that

   - If $x \neq 0$, then $\text{Ann}(x)$ is a **proper ideal** in R.
   - If for each maximal ideal $\mathbf{p}$ in R there exists some $r \in \text{Ann}(x)$, $r \notin \mathbf{p}$, then $x = 0$.

   b) Let R be an integral domain with field of fractions F. Show that

   - $T(M)$ is a submodule of M.
   - $T(M)$ is in the kernel of the map $M \to M \otimes_R F, \quad m \mapsto m \otimes 1$.
   - $T(M) = \{0\}$ if M is a **flat** R-module.

   c) True or false ? Prove the statement or give a counter example.

   For any R and an R-module M, $T(M)$ is a submodule of M.