METU MATHEMATICS DEPARTMENT
ALGEBRA II
SEPTEMBER 2012 - TMS EXAM

1. Let $f(x) = x^6 + 3 \in \mathbb{Q}[x]$, and let α be a root of $f(x)$ in $\mathbb{C}$.

a) Find the splitting field E of $f(x)$ over $\mathbb{Q}$.

b) Find the degree of the extension E over $\mathbb{Q}$.

c) Find the automorphism group G of E over $\mathbb{Q}$. Find the lattice of subgroups of G.

d) Choose one of nontrivial proper subgroups of G and find the intermediate field corresponding to this subgroup explicitly.

2. a) Prove that for a finite field F of characteristic p, the map $u \longmapsto u^p$ is an automorphism of F.

b) For every integer n, show that the map $u \longmapsto u^4 + u$ is an endomorphism of the additive group of the finite field $\mathbb{F}_{2^n}$, and determine the size of the kernel and image of this endomorphism.

3. Let R be a ring with 1, and let N be a submodule of an R-module M.

a) Prove that M is torsion if and only if N and M/N are both torsion.

b) Prove that if N and M/N are both torsion-free, then M is torsion-free. Give an example to show that the converse of this statement is false.

c) Prove that a free module over a PID is torsion-free. Give an eample to show that the converse of this statement is false.

4. Let R be a ring with 1 and let M, N be R-modules.

a) Prove that if M and N are both free, then $M \otimes_R N$ is free.

b) Let $f : M \to N$ be an R-homomorphism and let W be an R-module. Show that if f is surjective, then the induced map $f \otimes 1_W : M \otimes_R W \to N \otimes_R W$ is surjective. Give an example of M, N, W and an injective map $f : M \to N$ to show that the induced map $f \otimes 1_W$ is not injective.