METU Mathematics Department
Graduate Preliminary Examination
Algebra II, Fall 2014

1. Let A be an abelian group considered as a $\mathbb{Z}$-module. If A is finitely generated than show that $A \otimes_{\mathbb{Z}} A \cong A$ if and only if A is cyclic. Is the same statement true if A is not finitely generated?

2. Let $T : V \to W$ be a linear transformation of vector spaces over a field $\mathbb{F}$.

   (a) Show that T is injective if and only if $\{T(v_1), \ldots, T(v_n)\}$ is a linearly independent set in W for every linearly independent set $\{v_1, \ldots, v_n\}$ in V.

   (b) Show that T is surjective if and only if $\{T(x) : x \in X\}$ is a spanning set for W for some spanning set X for V.

   (c) Let $D : \mathbb{F}[x] \to \mathbb{F}[x]$ be the derivative map on polynomials, i.e. $D(f(x)) = f'(x)$, which is a linear transformation. Investigate if D is injective, surjective using the previous parts.

3. Let K be the splitting field of the polynomial $x^4 - x^2 - 1$ over $\mathbb{Q}$.

   (a) Show that $\sqrt{-1}$ is an element of K.

   (b) Show that the Galois group of K over $\mathbb{Q}$ is isomorphic to the dihedral group D_8.

   (c) Compute the lattice of subfields of K.

4. Let $\mathbb{F}_q$ be a finite field of order $q = p^n$ for some prime number p. Show that the set of subfields of $\mathbb{F}_q$ is linearly ordered (i.e. $L_1 \subseteq L_2$ or $L_2 \subseteq L_1$ for every pair of subfields.) if and only if n is a prime power.