1. Let $A$ be an abelian group considered as a $\mathbb{Z}$-module. If $A$ is finitely generated than show that $A \otimes_{\mathbb{Z}} A \cong A$ if and only if $A$ is cyclic. Is the same statement true if $A$ is not finitely generated?

2. Let $T : V \to W$ be a linear transformation of vector spaces over a field $F$.
   
   (a) Show that $T$ is injective if and only if $\{T(v_1), \ldots, T(v_n)\}$ is a linearly independent set in $W$ for every linearly independent set $\{v_1, \ldots, v_n\}$ in $V$.
   
   (b) Show that $T$ is surjective if and only if $\{T(x) : x \in X\}$ is a spanning set for $W$ for some spanning set $X$ for $V$.
   
   (c) Let $D : F[x] \to F[x]$ be the derivative map on polynomials, i.e. $D(f(x)) = f'(x)$, which is a linear transformation. Investigate if $D$ is injective, surjective using the previous parts.

3. Let $K$ be the splitting field of the polynomial $x^4 - x^2 - 1$ over $\mathbb{Q}$.
   
   (a) Show that $\sqrt{-1}$ is an element of $K$.
   
   (b) Show that the Galois group of $K$ over $\mathbb{Q}$ is isomorphic to the dihedral group $D_8$.
   
   (c) Compute the lattice of subfields of $K$.

4. Let $F_q$ be a finite field of order $q = p^n$ for some prime number $p$. Show that the set of subfields of $F_q$ is linearly ordered (i.e. $L_1 \subseteq L_2$ or $L_2 \subseteq L_1$ for every pair of subfields.) if and only if $n$ is a prime power.