1.a) Let \( R \) be a commutative ring with unity. A module \( P \) over \( R \) is called projective, if for every surjective module homomorphism \( f : N \to M \) and every module homomorphism \( g : P \to M \), there exists a homomorphism \( h : P \to N \) such that \( f \circ h = g \). Prove that every free \( R \)-module \( P \) is projective.

b) Show more generally that an \( R \)-module \( P \) is projective if and only if there is an \( R \)-module \( N \) such that \( P \oplus N \) is a free \( R \)-module.

c) Show that a finitely generated projective \( \mathbb{Z} \)-module \( P \) is indeed free. (This part of the question can be answered independently from the previous parts.)

2) Let \( A \) be a commutative ring with unity and \( M \) is a finitely generated \( A \)-module. Assume that \( f : M \to A^n \) is a surjective homomorphism. Show that \( \ker(f) \) is also finitely generated. (Hint: Choose a basis \( \{e_1, \ldots, e_n\} \) for \( A^n \), and let \( m_i \in M \) with \( f(m_i) = e_i \). Show that \( M \) is isomorphic to the direct sum of \( \ker(f) \) and the submodule generated by \( m_1, \ldots, m_n \).)

Is it true that a submodule of a finitely generated module is finitely generated?

3.a) Find the splitting field \( K \) of the polynomial \( f(x) = x^3 - 2 \in \mathbb{Q}[x] \).

b) Determine the Galois group of the extension \( K/\mathbb{Q} \).

c) Show that \( \sqrt{2} \) cannot be written as a \( \mathbb{Q} \)-linear combination of \( n \)th roots of unity for any positive integer \( n \).

4.a) Let \( f(x) \in \mathbb{Q}[x] \) be an irreducible polynomial and let \( G = \text{Gal}(K : \mathbb{Q}) \) be the Galois group of its splitting field \( K \). Considering \( G \) as a subgroup of \( S_5 \), the symmetric group on five letters, show that \( G \) contains a five cycle.

b) Assume further that exactly three roots of \( f(x) \) are real. Show that the Galois group \( G \) contains a transposition.

c) Is the polynomial \( f(x) \) solvable by radicals? (Hint: It is a fact that any subgroup of \( S_5 \) that contains a 5-cycle and a transposition is equal to \( S_5 \)).