METU MATHEMATICS DEPARTMENT
GRADUATE PRELIMINARY EXAMINATION
ALGEBRA II, SEPTEMBER 2015

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1.a) Let R be a commutative ring with unity. A module P over R is called projective, if for every surjective module homomorphism $f : N \to M$ and every module homomorphism $g : P \to M$, there exists a homomorphism $h : P \to N$ such that $f \circ h = g$. Prove that every free R-module P is projective.

b) Show more generally that an R-module P is projective if and only if there is an R-module N such that $P \oplus N$ is a free R-module.

c) Show that a finitely generated projective $\mathbb{Z}$-module P is indeed free. (This part of the question can be answered independently from the previous parts.)

2) Let A be a commutative ring with unity and M is a finitely generated A-module. Assume that $f : M \to A^n$ is a surjective homomorphism. Show that $\ker(f)$ is also finitely generated. (Hint: Choose a basis $\{e_1, \cdots, e_n\}$ for A^n, and let $m_i \in M$ with $f(m_i) = e_i$. Show that M is isomorphic to the direct sum of $\ker(f)$ and the submodule generated by $m_1, \cdots, m_n$)

Is it true that a submodule of a finitely generated module is finitely generated?

3.a) Find the splitting field K of the polynomial $f(x) = x^3 - 2 \in \mathbb{Q}[x]$.

b) Determine the Galois group of the extension $K/\mathbb{Q}$.

c) Show that $\sqrt[3]{2}$ cannot be written as a $\mathbb{Q}$-linear combination of n^{th} roots of unity for any positive integer n.

4.a) Let $f(x) \in \mathbb{Q}[x]$ be an irreducible polynomial and let $G = Gal(K : \mathbb{Q})$ be the Galois group of its splitting field K. Considering G as a subgroup of S_5, the symmetric group on five letters, show that G contains a five cycle.

b) Assume further that exactly three roots of $f(x)$ are real. Show that the Galois group G contains a transposition.

c) Is the polynomial $f(x)$ solvable by radicals? (Hint: It is a fact that any subgroup of S_5 that contains a 5-cycle and a transposition is equal to S_5).