METU Mathematics Department
Graduate Preliminary Examination
Algebra II, February 2016

1. Let R be an integral domain.

   (a) Show that free R-modules are torsion free.

   (b) Exhibit an R-module which is finitely generated and torsion free but not free.

   (c) Suppose that R is a principal ideal domain and let M be a finitely generated R-module. Show that M is torsion free if and only if M is free.

2. If $f : A \to A$ is an R-module homomorphism such that $f \circ f = f$, then show that

$$A = \operatorname{Ker}(f) \oplus \operatorname{Im}(f).$$

3. Let R be a commutative ring with identity. Let I and J be ideals of R. Prove that the R-module $(R/I) \otimes_R (R/J)$ is isomorphic to $R/(I+J)$.

4. Let K be the splitting field of the polynomial $f(x) = x^6 + 3$ over $\mathbb{Q}$. Show that the Galois group of K over $\mathbb{Q}$ is isomorphic to S_3.

5. Let F be a field.

   (a) Suppose that K is an algebraic extension of F and R is a ring such that $F \subseteq R \subseteq K$. Show that R is a field.

   (b) Suppose that $L = F(x)$, i.e. the field of rational functions over F. If $u \in L \setminus F$, then show that u is transcendental over F.