METU - Department of Mathematics
Graduate Preliminary Exam
Algebra - II
September 26, 2017

- Duration: 3 hours.
- Please write your solution for each question on a separate page.
- You can use any statement given below in the solution of any question even if you cannot prove that given statement.

Question 1.(3x12 pts.)

a) Show that every unitary left $R$-module $A$ over a ring $R$ with identity has a free-resolution, that is, show that there exists an infinite exact sequence

$$
\cdots \rightarrow F_n \rightarrow_{s_n} F_{n-1} \rightarrow_{s_{n-1}} \cdots \rightarrow F_1 \rightarrow_{s_1} A \rightarrow 0
$$

of left $R$-modules such that each $F_i$ ($i \geq 1$) is a free $R$-module (Consider 0 as a free $R$-module with no generators).

b) For a right ideal $I$ of a ring $R$ with identity and a left $R$-module $B$ show that there is a group isomorphism $(R/I) \otimes B \cong B/IP$ where $IP$ is the subgroup of $B$ generated by all elements of the form $rb$ for $r \in I$ and $b \in B$.

c) Let $R$ be a principal ideal domain and let $p, q, r \in R$ be three distinct primes. Assume that $A$ is a unitary $R$-module generated by two elements as an $R$-module but $A \not\cong RA$ for any $a \in A$. If $\{r \in R | ra = 0 \text{ for all } a \in A\} = (p^a q^b r^c)$ as an ideal of $R$, then list all distinct $R$-modules $A$ (satisfying these conditions) up to isomorphism.

Question 2.(3x7 pts.) Let $F$ be the splitting field of $g(x) = x^4 - 2$ over $\mathbb{Q}$.

a) Determine $F$ as a subfield of $\mathbb{C}$. What is $[F : \mathbb{Q}]$?

b) What is the order of the Galois group $\text{Aut}_\mathbb{Q}F$? To which familiar finite group is $\text{Aut}_\mathbb{Q}F$ isomorphic?

c) How many intermediate fields are there for the field extension $F$ over $\mathbb{Q}$?

Question 3.(7 pts.) Show that if $u$ is algebraic over a field $K$ such that $[K(u) : K]$ is odd, then $K(u) = K(u^2)$.

Question 4.(2x9 pts.)

a) For any prime $p$ show that the finite field $F_{p^n}$ is a field extension of $F_{p^n}$ if and only if $n|s$.

b) Show by using the field extensions mentioned in part (a) that $K = \bigcup_{n \geq 1} F_{p^n}$ is a field and $K$ is the algebraic closure of $F_p = \mathbb{Z}_p$. (Note that in $K$ we consider any $F_{p^n}$ as a subfield of $F_{p^n}$ whenever $n|s$).

Question 5.(2x9 pts.)

a) For field extensions $K \subset E \subset F$, if $F$ is separable over $K$, show that $F$ is separable over $E$.

b) For field extensions $K \subset E \subset F$, assume that $E$ is a normal extension over $K$. Show that any $\tau \in \text{Aut}_K F$ restricts to a $K$-automorphism of $E$. 