METU - Department of Mathematics
Graduate Preliminary Exam
Algebra - II
September 26, 2017

- Duration: 3 hours.
- Please write your solution for each question on a separate page.
- You can use any statement given below in the solution of any question even if you cannot prove that given statement.

Question 1.(3x12 pts.)a) Show that every unitary left R-module A over a ring R with identity has a free-resolution, that is, show that there exists an infinite exact sequence

$$\cdots F_n \xrightarrow{g_n} F_{n-1} \xrightarrow{g_{n-1}} \cdots F_1 \xrightarrow{g_1} A \to 0$$

of left R-modules such that each F_i $(i \geq 1)$ is a free R-module (Consider 0 as a free R-module with no generators).
b) For a right ideal I of a ring R with identity and a left R-module B show that there is a group isomorphism $(R/I) \otimes_R B \cong B/IB$ where IB is the subgroup of B generated by all elements of the form rb for $r \in I$ and $b \in B$.
c) Let R be a principal ideal domain and let $p, q, r \in R$ be three distinct primes. Assume that A is a unitary R-module generated by two elements as an R-module but $A \neq Rz$ for any $z \in A$. If $\{r \in R | ra = 0 \text{ for all } a \in A\} = (p^2q^2r)$ as an ideal of R, then list all distinct R-modules A (satisfying these conditions) up to isomorphism.

Question 2.(3x7 pts.) Let F be the splitting field of $g(x) = x^4 - 2$ over $\mathbb{Q}$.
a) Determine F as a subfield of $\mathbb{C}$. What is $[F : \mathbb{Q}]$?
b) What is the order of the Galois group $Aut_{\mathbb{Q}}F$? To which familiar finite group is $Aut_{\mathbb{Q}}F$ isomorphic?
c) How many intermediate fields are there for the field extension F over $\mathbb{Q}$?

Question 3.(7 pts.) Show that if u is algebraic over a field K such that $[K(u) : K]$ is odd, then $K(u) = K(u^2)$.

Question 4.(2x9 pts.)a) For any prime p show that the finite field F_{p^s} is a field extension of F_{p^n} if and only if $n|s$.
b) Show by using the field extensions mentioned in part (a) that $K = \bigcup_{n\geq 1} F_{p^n}$ is a field and K is the algebraic closure of $F_p = \mathbb{Z}_p$.(Note that in K we consider any F_{p^n} as a subfield of F_{p^s} whenever $n|s$).

Question 5.(2x9 pts.)a) For field extensions $K \subset E \subset F$, if F is separable over K, show that F is separable over E.
b) For field extensions $K \subset E \subset F$, assume that E is a normal extension over K. Show that any $\tau \in Aut_K F$ restricts to a K-automorphism of E.