

METU - Department of Mathematics  
Graduate Preliminary Exam  
Algebra - II  
September 26, 2017

- Duration: 3 hours.
- Please write your solution for each question on a separate page.
- You can use any statement given below in the solution of any question even if you cannot prove that given statement.

**Question 1.(3x12 pts.)**a) Show that every unitary left  $R$ -module  $A$  over a ring  $R$  with identity has a free-resolution, that is, show that there exists an infinite exact sequence

$$\cdots F_n \xrightarrow{g_n} F_{n-1} \xrightarrow{g_{n-1}} \cdots F_1 \xrightarrow{g_1} A \rightarrow 0$$

of left  $R$ -modules such that each  $F_i$  ( $i \geq 1$ ) is a free  $R$ -module (Consider 0 as a free  $R$ -module with no generators).

b) For a right ideal  $I$  of a ring  $R$  with identity and a left  $R$ -module  $B$  show that there is a group isomorphism  $(R/I) \otimes_R B \cong B/IB$  where  $IB$  is the subgroup of  $B$  generated by all elements of the form  $rb$  for  $r \in I$  and  $b \in B$ .

c) Let  $R$  be a principal ideal domain and let  $p, q, r \in R$  be three distinct primes. Assume that  $A$  is a unitary  $R$ -module generated by two elements as an  $R$ -module but  $A \neq Rz$  for any  $z \in A$ . If  $\{r \in R \mid ra = 0 \text{ for all } a \in A\} = (p^2q^2r)$  as an ideal of  $R$ , then list all distinct  $R$ -modules  $A$  (satisfying these conditions) up to isomorphism.

**Question 2.(3x7 pts.)** Let  $F$  be the splitting field of  $g(x) = x^4 - 2$  over  $\mathbb{Q}$ .

- a) Determine  $F$  as a subfield of  $\mathbb{C}$ . What is  $[F : \mathbb{Q}]$ ?
- b) What is the order of the Galois group  $\text{Aut}_{\mathbb{Q}} F$ ? To which familiar finite group is  $\text{Aut}_{\mathbb{Q}} F$  isomorphic?
- c) How many intermediate fields are there for the field extension  $F$  over  $\mathbb{Q}$ ?

**Question 3.(7 pts.)** Show that if  $u$  is algebraic over a field  $K$  such that  $[K(u) : K]$  is odd, then  $K(u) = K(u^2)$ .

**Question 4.(2x9 pts.)**a) For any prime  $p$  show that the finite field  $F_{p^s}$  is a field extension of  $F_{p^n}$  if and only if  $n|s$ .

b) Show by using the field extensions mentioned in part (a) that  $K = \bigcup_{n \geq 1} F_{p^n}$  is a field and  $K$  is the algebraic closure of  $F_p = \mathbb{Z}_p$ . (Note that in  $K$  we consider any  $F_{p^n}$  as a subfield of  $F_{p^s}$  whenever  $n|s$ ).

**Question 5.(2x9 pts.)**a) For field extensions  $K \subset E \subset F$ , if  $F$  is separable over  $K$ , show that  $F$  is separable over  $E$ .

b) For field extensions  $K \subset E \subset F$ , assume that  $E$  is a normal extension over  $K$ . Show that any  $\tau \in \text{Aut}_K F$  restricts to a  $K$ -automorphism of  $E$ .