Graduate Preliminary Examination

Algebra II

18.2.2004: 3 hours

Problem 1. Let q be a prime power. Suppose f is an irreducible polynomial of degree m over $\mathbb{F}_q$, and let α be a root of f.

(a) Prove that $\alpha \in \mathbb{F}_{q^m}$.

(b) Prove that α^{q^n} is a root of f in F_{q^m} for all integers n.

(c) Prove that $\alpha, \alpha^q, \alpha^{q^2}, \ldots, \alpha^{q^{m-1}}$ are distinct roots of f.

Problem 2. Suppose K is an algebraic extension of a field F. Prove that the following are equivalent:

- K is algebraically closed.
- For every algebraic extension L of F, there is an F-monomorphism from L to K.

Problem 3. Let M be a module over a ring R. An element x of M is called **torsion** if $rx = 0$ for some non-zero r in R. Let $T(M)$ be the set of torsion elements of M.

(a) Prove that, if R is an integral domain, then $T(M)$ is a submodule of M, and $M/T(M)$ has no torsion elements.

(b) Find an example where $T(M)$ is not a submodule of M.

Problem 4. Let R be a commutative ring with identity, and let M be a non-zero (unitary) R-module. If $m \in M$, let

$$\operatorname{ord} m = \{r \in R : rm = 0\},$$

and define

$$\mathcal{F} = \{\operatorname{ord} m : m \in M \setminus \{0\}\}.$$

Then $\mathcal{F}$ is partially ordered by $\subseteq$.

(a) Prove that $\operatorname{ord} m$ is an ideal of R.

(b) Prove that every maximal element of $\mathcal{F}$ is a prime ideal.