Problem 1. Let \( q \) be a prime power. Suppose \( f \) is an irreducible polynomial of degree \( m \) over \( \mathbb{F}_q \), and let \( \alpha \) be a root of \( f \).
(a) Prove that \( \alpha \in \mathbb{F}_{q^m} \).
(b) Prove that \( \alpha^n \) is a root of \( f \) in \( \mathbb{F}_{q^n} \) for all integers \( n \).
(c) Prove that \( \alpha, \alpha^n, \alpha^{2n}, \ldots, \alpha^{(m-1)n} \) are distinct roots of \( f \).

Problem 2. Suppose \( K \) is an algebraic extension of a field \( F \). Prove that the following are equivalent:

- \( K \) is algebraically closed.
- For every algebraic extension \( L \) of \( F \), there is an \( F \)-isomorphism from \( L \) to \( K \).

Problem 3. Let \( M \) be a module over a ring \( R \). An element \( x \) of \( M \) is called torsion if \( rx = 0 \) for some non-zero \( r \) in \( R \). Let \( T(M) \) be the set of torsion elements of \( M \).

(a) Prove that, if \( R \) is an integral domain, then \( T(M) \) is a submodule of \( M \), and \( M/T(M) \) has no torsion elements.
(b) Find an example where \( T(M) \) is not a submodule of \( M \).

Problem 4. Let \( R \) be a commutative ring with identity, and let \( M \) be a non-zero (unitary) \( R \)-module. If \( m \in M \), let
\[
\text{ord } m = \{ r \in R : rm = 0 \},
\]
and define
\[
\mathcal{F} = \{ \text{ord } m : m \in M \setminus \{0\} \}.
\]
Then \( \mathcal{F} \) is partially ordered by \( \subseteq \).

(a) Prove that \( \text{ord } m \) is an ideal of \( R \).
(b) Prove that every maximal element of \( \mathcal{F} \) is a prime ideal.