Duration: 180 min.
1. Let \( \alpha \) be the real positive 16th root of 3 and consider the chain of intermediate fields
\[ \mathbb{Q} \subseteq \mathbb{Q}(\alpha^8) \subseteq \mathbb{Q}(\alpha^4) \subseteq \mathbb{Q}(\alpha^2) \subseteq \mathbb{Q}(\alpha) = F. \]
   a) Compute the degrees of these five intermediate fields over \( \mathbb{Q} \) and conclude that these fields are all distinct.
   b) Show that every intermediate field between \( \mathbb{Q} \) and \( F \) is one of the above. (Hint: If \( \mathbb{Q} \subseteq K \subseteq F \), consider the constant term of the minimal polynomial of \( \alpha \) over \( K \)).

2. Let \( p \) be a prime number and let \( w_p = e^{2\pi i/p} \) be the \( p \)th root of 1 in \( \mathbb{C} \).
   a) Show that \( \text{Gal} (\mathbb{Q}(w_p)/\mathbb{Q}) \) is isomorphic to the multiplicative group \( \mathbb{Z}_p^* \).
   b) Let \( F \) be a field containing \( w_p \) and let \( a \) be an element of \( F \) which is not the \( p \)th power of any element of \( F \). Show that if \( E \) is the splitting field of the polynomial \( x^p - a \in F[x] \), then \( \text{Gal} (E/F) \) is isomorphic to the additive group \( \mathbb{Z}_p \).
   c) If \( K \) is the splitting field of \( x^p - 2 \in \mathbb{Q}[x] \), show that \( |K : \mathbb{Q}| = p(p - 1) \).

3. Let \( R \) be a ring. Recall that an \( R \)-module \( P \) is called projective if for every \( R \)-module epimorphism \( f : A \to B \) and every \( R \)-module homomorphism \( g : P \to B \), there exists an \( R \)-module homomorphism \( h : P \to A \) such that \( fh = g \).
   a) Let \( P \) be an \( R \)-module for a ring \( R \). Show that if there is a free \( R \)-module \( F \) and an \( R \)-module \( K \) such that \( F \cong K \otimes P \), then \( P \) is projective. (You may use the fact that every free module is projective).
   b) Let \( R \) be a commutative ring. Suppose that \( R \)-modules \( P \) and \( Q \) are projective. Show that \( P \otimes_R Q \) is projective.

4. Let \( R \) be a ring with unity and suppose that \( R \) can be written as the sum \( R = \sum_{i=1}^{m} I_i \), where \( I_i \) are finitely many (two-sided) ideals of \( R \) satisfying \( I_i \cap I_j = 0 \) whenever \( i \neq j \).
   a) Prove that, for every simple right \( R \)-module \( M \), there exists a unique subscript \( k \) such that \( M I_k \neq 0 \)
   b) Show that if \( i \neq j \), then every right \( R \)-module homomorphism \( \theta : I_i \to I_j \) is the zero map.