1. Let $R$ be a commutative ring with identity 1 and let $Q$ be an injective $R$-module. If $L \xrightarrow{\alpha} M \xrightarrow{\beta} N$ is an exact sequence of $R$-modules and $R$-homomorphisms with the property that $f \circ \alpha = 0$ for an $R$-homomorphism $f : M \rightarrow Q$, show that there is an $R$-homomorphism $g : N \rightarrow Q$ with $g \circ \beta = f$.

2. A nonzero left module $M$ (over some ring) is called
- **simple**, if $M$ has no proper nonzero submodule;
- **complemented**, if every submodule of $M$ is a direct summand of $M$ (that is, for every submodule $A$ of $M$, there is a submodule $B$ of $M$ such that $M = A \oplus B$, which means $M = A + B$ and $A \cap B = 0$).

   (a) Give an example of a simple module.
   (b) Give an example of a complemented module that is not simple.
   (c) Show that every nonzero submodule of a complemented module is complemented.
   (d) Show that every complemented module has a simple submodule.

3. Suppose $K$, $L$, and $M$ are fields, and $K \subseteq L \subseteq M$. Prove or disprove the following statements.
   (a) If $M/L$ and $L/K$ are normal, then so is $M/K$.
   (b) If $M/K$ is normal, then so is $M/L$.
   (c) If $M/L$ is normal, then so is $M/K$.
   (d) $(K, +) \not\cong (K^*, \cdot)$.

4. Consider the polynomial $f(x) = x^5 - 6x + 3 \in \mathbb{Q}[x]$
   (a) Using Eisenstein’s criterion, prove that $f$ is irreducible over $\mathbb{Q}$.
   (b) Let $E$ be the splitting field of $f$. Show that there exists $\sigma \in Gal(E/\mathbb{Q})$ of order 5.
   (c) Prove the following:
      There exists $\tau \in Gal(E/\mathbb{Q})$ of order 2 and hence $Gal(E/\mathbb{Q}) \cong S_5$.
      (Hint: You may assume that $f(x)$ has exactly one pair of complex conjugate roots.)
   (d) Is $f(x)$ solvable by radicals over $\mathbb{Q}$? Why?