GRADUATE PRELIMINARY EXAMINATION
ALGEBRA II, FEBRUARY 2013

FEBRUARY 14, 2013

1.a. Show that $f(x) = x^3 + x + 1 \in \mathbb{F}_2[x]$ is the unique irreducible polynomial of degree three so that sum of its roots in a splitting field is equal to zero, where $\mathbb{F}_2$ denotes the field of two elements.

1.b. Show that $E = \mathbb{F}_2[x]/(f(x))$ is the splitting field of $f(x)$, by finding all zeros of $f(x) = x^3 + x + 1$ in E.

1.c. Show that the extension $\mathbb{F}_2 \subseteq E$ is Galois, by determining its Galois group. Describe the action of the elements of the Galois group.

2.a. Construct a Galois field extension K of the field of rational numbers, so that $\mathbb{Q} \subseteq K \subseteq \mathbb{R}$ and the Galois group is isomorphic to the Klein four group $\mathbb{Z}_2 \times \mathbb{Z}_2$.

2.b. Determine the intermediate fields of this extension.

2.c. Show that $\mathbb{Q}(\sqrt[4]{2})$ is a degree four extension of $\mathbb{Q}$, which is not Galois.

3.a. Let M denote the free $\mathbb{Z}$-module $\mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z}$. Determine the quotient module M/N, where N is generated by the vectors $(1, 2, 3)$, $(0, -2, 5)$, $(2, 0, 8)$ and $(0, 1, 2)$. (Use Smith Normal From.) Calculate the number elements of the quotient module if it is finite.

3.b. Let $\mathbb{F}$ be a field, $R = \mathbb{F}[x]$ the polynomial ring over $\mathbb{F}$, and $M = R \oplus \cdots \oplus R$ the free R-module of rank n, for some positive fixed integer n. For any $n \times n$-matrix A with entries in the field $\mathbb{F}$, consider the assocaited matrix $A - xI_{n\times n}$, with entries in the ring R, where $I_{n\times n}$ is the $n \times n$-identity matrix. Let N_A be the submodule of M generated by the rows of $A - xI_{n\times n}$. Calculate the quotient modules, M/N_{A_i}, using Smith Normal Form again, for the matrices

$$A_1 = \begin{pmatrix} 2 & 0 & 0 \\ 1 & 2 & 0 \\ 0 & 1 & 2 \end{pmatrix} \qquad \text{and} \qquad A_2 = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 1 & 2 \end{pmatrix},$$

with entries in the field of rational numbers (i.e., we take $\mathbb{F} = \mathbb{Q}$ and $n = 3$). Are the quotient modules isomorphic? Relate the quotient modules to the characteristic and the minimal polynomial of the matrices.

4. Let R be an integral domain and M an R-module so that for any $r \neq 0 \in R$ and $m \neq 0 \in M$, we have $rm \neq 0$. The R-module M is called divisible if for each $m \in M$ and nonzero element $r \in R$ there exists an element $m' \in M$ such that $m = rm'$.

4.a. Prove that the direct sum of two divisible R-modules is also divisible.

4.b. An R-module M is called injective, if whenever $i : M \to N$ is an embedding of R-modules and $\phi : M \to L$ is an R-module homomorphism then there is an R-module homomorphism $\Phi : N \to L$ so that $\phi(m) = (\Phi \circ i)(m)$, for all $m \in M$. Prove that an injective R-module is divisible.