1.a. Show that \( f(x) = x^3 + x + 1 \in \mathbb{F}_2[x] \) is the unique irreducible polynomial of degree three so that sum of its roots in a splitting field is equal to zero, where \( \mathbb{F}_2 \) denotes the field of two elements.

1.b. Show that \( E = \mathbb{F}_2[x]/(f(x)) \) is the splitting field of \( f(x) \), by finding all zeros of \( f(x) = x^3 + x + 1 \) in \( E \).

1.c. Show that the extension \( \mathbb{F}_2 \subseteq E \) is Galois, by determining its Galois group. Describe the action of the elements of the Galois group.

2.a. Construct a Galois field extension \( K \) of the field of rational numbers, so that \( \mathbb{Q} \subseteq K \subseteq \mathbb{R} \) and the Galois group is isomorphic to the Klein four group \( \mathbb{Z}_2 \times \mathbb{Z}_2 \).

2.b. Determine the intermediate fields of this extension.

2.c. Show that \( \mathbb{Q}(\sqrt{2}) \) is a degree four extension of \( \mathbb{Q} \), which is not Galois.

3.a. Let \( M \) denote the free \( \mathbb{Z} \)-module \( \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z} \). Determine the quotient module \( M/N \), where \( N \) is generated by the vectors \((1, 2, 3), (0, -2, 5), (2, 0, 8) \) and \((0, 1, 2)\). (Use Smith Normal Form.) Calculate the number elements of the quotient module if it is finite.

3.b. Let \( F \) be a field, \( R = F[x] \) the polynomial ring over \( F \), and \( M = R \oplus \cdots \oplus R \) the free \( R \)-module of rank \( n \), for some positive fixed integer \( n \). For any \( n \times n \)-matrix \( A \) with entries in the field \( F \), consider the associated matrix \( A - xI_{n \times n} \), with entries in the ring \( R \), where \( I_{n \times n} \) is the \( n \times n \)-identity matrix. Let \( N_A \) be the submodule of \( M \) generated by the rows of \( A - xI_{n \times n} \). Calculate the quotient modules, \( M/N_A \), using Smith Normal Form again, for the matrices

\[
A_1 = \begin{pmatrix}
2 & 0 & 0 \\
1 & 2 & 0 \\
0 & 1 & 2
\end{pmatrix}
\quad \text{and} \quad
A_2 = \begin{pmatrix}
2 & 0 & 0 \\
0 & 2 & 0 \\
0 & 1 & 2
\end{pmatrix},
\]

with entries in the field of rational numbers (i.e., we take \( F = \mathbb{Q} \) and \( n = 3 \)). Are the quotient modules isomorphic? Relate the quotient modules to the characteristic and the minimal polynomial of the matrices.
4. Let $R$ be an integral domain and $M$ an $R$-module so that for any $r \neq 0 \in R$ and $m \neq 0 \in M$, we have $rm \neq 0$. The $R$-module $M$ is called divisible if for each $m \in M$ and nonzero element $r \in R$ there exists an element $m' \in M$ such that $m = rm'$.

4.a. Prove that the direct sum of two divisible $R$-modules is also divisible.

4.b. An $R$-module $M$ is called injective, if whenever $i : M \to N$ is an embedding of $R$-modules and $\phi : M \to L$ is an $R$-module homomorphism then there is an $R$-module homomorphism $\Phi : N \to L$ so that $\phi(m) = (\Phi \circ i)(m)$, for all $m \in M$. Prove that an injective $R$-module is divisible.