METU Mathematics Department
Graduate Preliminary Examination
Algebra II, January 2014

1. Let $I = (2, x)$ be the ideal generated by 2 and x in the ring $R = \mathbb{Z}[x]$. Note that the ring $\mathbb{Z}/2\mathbb{Z} \cong R/I$ is naturally an R-module.
   - Show that the map $\phi : I \times I \to \mathbb{Z}/2\mathbb{Z}$ defined by
   $$\phi(a_0 + a_1 x + \ldots + a_n x^n, b_0 + b_1 x + \ldots + b_m x^m) = \frac{a_0}{2} b_1 \pmod 2$$
   is R-bilinear.
   - Show that $2 \otimes x - x \otimes 2$ is nonzero in $I \otimes_R I$.
2. Let I_n be the identity matrix of dimension n.
   - Prove that there is no 3×3 matrix A over $\mathbb{Q}$ such that $A^8 = I_3$ but $A^4 \neq I_3$.
   - Write down a 4×4 matrix B over $\mathbb{Q}$ such that $B^8 = I_4$ and $B^4 \neq I_4$.
3. Let $\mathbb{F}_q$ be a finite field with q elements and let $K/\mathbb{F}_q$ be a quadratic extension.
   - For any $\alpha \in K$, show that $\alpha^{q+1} \in \mathbb{F}_q$.
   - Show that every element of $\mathbb{F}_q$ is of the form β^{q+1} for some $\beta \in K$.
4. Let p be an odd prime and let $L = \mathbb{Q}(\zeta_p)$ be the p-th cyclotomic field.
   - Show that L has a unique subfield K such that $[K : \mathbb{Q}] = 2$.
   - If $p = 5$, then find an element $\alpha \in L$ such that $L = K(\alpha)$ and $\alpha^2 \in K$.
   - If $p \geq 7$, then show that there is no $\alpha \in L$ such that $L = K(\alpha)$ and $\alpha^{(p-1)/2} \in K$.