(1) (25 Points) If $T : V \to V$ is a linear transformation on a vector space $V$ over a field $F$, then $V$ can be made into an $F[t]$-module by setting $t \cdot v = T(v)$ for any $v \in V$. For each of the following transformations $T$, classify all $F[t]$-submodules of $V$.

(a) $V = \mathbb{R}^2$ and $T(x, y) = (-y, x)$.
(b) $V = \mathbb{R}^3$ and $T(x, y, z) = (z, x, y)$.
(c) $V = \mathbb{R}^3$ and $T(x, y, z) = (y, x, 0)$.

(2) (25 Points) State the definitions of projective and injective modules. Using the definition, show that a nonzero finite abelian group is neither a projective nor an injective $\mathbb{Z}$-module.

(3) (25 Points) Let $R$ be an integral domain. An element $m$ of an $R$-module $M$ is called a torsion element if there exists a nonzero element $r \in R$ such that $rm = 0$.

(a) If $I \subseteq R$ is a principal ideal, then prove that the $R$-module $I \otimes_R I$ has no torsion element other than zero.
(b) In particular, consider $R = \mathbb{Z}[x, y]$ and $I = (x, y)$. Show that the $R$-module $I \otimes_R I$ has a nonzero torsion element.

(4) (25 Points) Let $L$ be the splitting field of $f(x) = x^6 + 3$ over $\mathbb{Q}$. Show that the Galois group $G = \text{Gal}(L/\mathbb{Q})$ is isomorphic to $S_3$. Determine all proper subfields $\mathbb{Q} \subsetneq K \subsetneq L$ by using the fundamental theorem of Galois theory. For each $K$, find an explicit element $\alpha \in L$ so that $K = \mathbb{Q}(\alpha)$. 