Complex Analysis

1. Evaluate \( \int_0^\infty e^{-x^2} \cos(x^2) \, dx \) using complex integration along the given contour.
   (Hint: \( \int_0^\infty e^{-u^2} \, du = \frac{\sqrt{\pi}}{2} \)).

2. Let \( f : \mathbb{C}^* \to \mathbb{C} \) be an analytic map such that for all \( z \in \mathbb{C} \) the set \( f^{-1}(z) \) is finite (if not empty). Show that
   (i) \( z = 0 \) is not an essential singularity of \( f \).
   (ii) If \( f \) is bounded in some deleted neighborhood of 0, then \( f \) is a polynomial.

3. Recall that the analytic automorphisms of the unit disk \( D \) are given by linear fractional transformations of the form \( z \mapsto e^{i\theta} \frac{z - \alpha}{1 - \overline{\alpha} z} \) for some \( \theta \in [0, 2\pi) \) and \( \alpha \in D \).
   a) Using this fact prove that the analytic automorphisms of the upper half-plane \( \mathcal{H} \) are given by (special) linear fractional transformations.
   b) Show that the map \( \Omega = \{ z : 0 < \arg(z) < \frac{\pi}{2} \} \to \mathcal{H} \), \( z \mapsto z^2 \) is an analytic isomorphism.
   Deduce that if \( g \in Aut(\Omega) \), then there exists a linear fractional transformation \( T \) such that \( g(z) = \sqrt{T(z^2)} \) for a suitable branch of the square root function (which branch?).
   c) Show that there exists no linear fractional transformation which maps \( \Omega \) isomorphically onto \( D \).

4. Let \( f : \mathbb{C} \to \mathbb{C} \) be a rational function such that \( |f(z)| = 1 \) if \( |z| = 1 \). Prove that there exist \( c \in \mathbb{C} \), \( c \neq 0 \) and \( \alpha_1, \ldots, \alpha_n \in \mathbb{C} \), \( |\alpha_i| \neq 0,1 \) and \( m \in \mathbb{Z} \) such that
   \[
   f(z) = cz^m \prod_{i=1}^{n} \frac{z - \alpha_i}{1 - \overline{\alpha_i} z} .
   \]