METU - Mathematics Department
Graduate Preliminary Exam-Fall 2007

Complex Analysis

1. Evaluate $\int_0^\infty e^{-x^2}\cos(x^2)dx$ using complex integration along the given contour.
   (Hint: $\int_0^\infty e^{-u^2}du = \frac{\sqrt{\pi}}{2}$).

2. Let $f : \mathbb{C}^* \to \mathbb{C}$ be an analytic map such that for all $z \in \mathbb{C}$ the set $f^{-1}(z)$ is finite (if not empty). Show that

   (i) $z = 0$ is not an essential singularity of f.

   (ii) If f is bounded in some deleted neighborhood of 0, then f is a polynomial.

3. Recall that the analytic automorphisms of the unit disk D are given by linear fractional transformations of the form $z \mapsto e^{i\theta}\frac{z-\alpha}{1-\overline{\alpha}z}$ for some $\theta \in [0, 2\pi)$ and $\alpha \in$ D.

   a) Using this fact prove that the analytic automorphisms of the upper half-plane $\mathcal{H}$ are given by (special) linear fractional transformations.

   b) Show that the map $\Omega = \{z : 0 < \arg(z) < \frac{\pi}{2}\} \to \mathcal{H},\ z \mapsto z^2$ is an analytic isomorphism.

   Deduce that if $g \in Aut(\Omega)$, then there exists a linear fractional transformation T such that $g(z) = \sqrt{T(z^2)}$ for a suitable branch of the square root function (which branch ?).

   c) Show that there exists no linear fractional transformation which maps Ω isomorphically onto D.

4. Let $f : \mathbb{C} \to \mathbb{C}$ be a rational function such that $|f(z)| = 1$ if $|z| = 1$. Prove that there exist $c \in \mathbb{C},\ c \neq 0$ and $\alpha_1, ..., \alpha_n \in \mathbb{C}, |\alpha_i| \neq 0, 1$ and $m \in \mathbb{Z}$ such that
   $$f(z) = cz^m \prod_1^n \frac{z-\alpha_i}{1-\overline{\alpha}_i z}.$$