TMS

Fall 2009

Complex Analysis

I. a) Find the conformal map

$$T : \mathcal{H} = \{z : Imz > 0\} \rightarrow D = \{z : |z| < 1\}$$

which satisfies

$T(i) = 0$

$T(1) = 1$

(Hint: Consider symmetry).

b) Map the region

$$\Omega = \{z : 0 < Re(z) < 2\}$$

conformally onto D.

(Hint: First map Ω into $\mathcal{H}$).

2. a) Formulate **precisely**, the Cauchy theorem for complex integration and its partial converse (the Morera's Theorem)

b) Using Morera's Theorem, prove that every function f which is continuous in the open disk D and analytic on $D - \{1/2\}$ is analytic on D.

3. Let f be a meromorphic function in $\mathbb{C}$ whose poles all lie on the line $y = x$ (for example $f(x) = \tan(\frac{z}{1+i})$) and for $r \in \mathbb{R}_{+}$, let $C(r)$ be the circle $|z| = r$.

a) For which circles $C(r)$, is $\int_{C(r)} f(z)dz$ defined?

b) Show that the formula

$$F(r) = \int_{C(r)} f(z)dz$$

defines a function $(0, \infty) - D \to \mathbb{C}$ where D is a discrete subset of $(0, \infty)$.

c) Show that if $f(z)$ has only finitely many poles in $\mathbb{C}$, then there exists some $R > 0$ such that F defines a constant function on (R, ∞).

4. Consider the open disc $D(0,1)$. Let $a, b \in D(0,1)$ be two distinct points.

a) Write the most general analytic automorphism

$\sigma : D(0,1) \to D(0,1)$

such that $\sigma(a) = 0$.

b) Show that $\mathrm{Aut}(D(0,1))$ acts transitively on $D(0,1)$, by writing down

$$\tau \in AutD((0,1))$$

such that $\tau(a) = b$.

c) True or false? why?

If $\Omega \subset \mathbb{C}$ is any **simply connected** region, and if $a, b \in \Omega$, then there exists an analytic automorphism $\psi : \Omega \to \Omega$ such that $\psi(a) = b$.