1. For \( R > 0 \), let \( \Gamma_R \) be the counterclockwise oriented closed curve obtained by joining the following paths, in the given order:

\[
\begin{align*}
\Gamma_{1,R} &: [0, R] \to \mathbb{C} \text{ defined by } \Gamma_{1,R}(t) = t \\
\Gamma_{2,R} &: [0, \pi/2] \to \mathbb{C} \text{ defined by } \Gamma_{2,R}(t) = Re^{it} \\
\Gamma_{3,R} &: [0, R] \to \mathbb{C} \text{ defined by } \Gamma_{3,R}(t) = (R - t)i.
\end{align*}
\]

Consider the analytic function \( f : \mathbb{C} - \{0\} \to \mathbb{C} \) defined by

\[ f(z) = \frac{e^{-z} - e^{iz}}{z} \]

for each \( z \neq 0 \).

(A) Show that \( z = 0 \) is a removable singularity of \( f(z) \).

(B) Prove that

\[ \int_{\Gamma_R} f(z)\,dz = 0. \]

(C) Prove that

\[ \lim_{R \to \infty} \int_{\Gamma_{2,R}} f(z)\,dz = 0. \]

(D) Prove that

\[ \int_0^\infty \frac{e^{-x} - \cos x}{x} \,dx = 0. \]
2. Consider
\[ \Omega = \{ z \in \mathbb{C} \mid \Im(z) > 0 \} \]
and
\[ \Delta = \{ z \in \mathbb{C} \mid |z| < 1 \} \, . \]

(A) Given any \( b \in \Omega \) prove that the map \( \Psi = \Psi_b : \mathbb{C} - \{\bar{b}\} \to \mathbb{C} \) defined for each \( z \neq \bar{b} \) by
\[ \Psi(z) = \frac{z - b}{z - \bar{b}} \]
maps \( \Omega \) onto \( \Delta \) bijectively.

(B) If \( f : \Omega \to \mathbb{C} \) is analytic and satisfies \( f(\Omega) \subseteq \Omega \), prove that
\[ \frac{|f(z) - f(a)|}{|f(z) - f(a)|} \leq \frac{|z - a|}{|z - \bar{a}|} \]
for any \( z, a \in \Omega, z \neq a \).\(^1\)

(C) Deduce that
\[ |f'(z)| \leq \left| \frac{\Im(f(z))}{\Im(z)} \right| \]
for any \( z \in \Omega \).

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\(^1\) Consider \( g = \Psi_{f(a)} \circ f \circ \Psi_{a}^{-1} \).
3. Let $\Omega \subset \mathbb{R}^2 \cong \mathbb{C}$ be an open connected region and let

$$f = (u, v) : \Omega \to \mathbb{C}$$

be a **non-constant differentiable** function. Consider the set

$$Z_d f = \{ z = (x, y) \in \Omega : \det(df(x, y)) = 0 \}.$$

a) Give an example $f(x, y)$ for which $Z_d f$ is not a discrete subset of $\Omega$.

b) Suppose that $f(z)$ is an analytic function.

- Show that $Z_d f$ is discrete.
- Suppose that $z_0 \in Z_d f$ and that $f(z_0) = 0$. Show that there exists an integer $n > 1$ such that the function $g(z) = 1/f(z)$ maps a neighborhood of $z_0$ analytically onto a neighborhood of $\infty$ in an $n$-to-one manner.
- Show that if $\operatorname{Res}(g; z_0) = 0$ then $g(z)$ is the derivative of a function meromorphic around $z_0$.

4. a) Show that if $f(z)$ is a non-constant entire periodic function, then $f(z)$ has an essential singularity at $\infty$.

b) Does there exist a non-constant entire doubly periodic function? Explain.

(Recall that a meromorphic function $f(z)$ is said to be doubly periodic with periods $w_1, w_2$ if $w_1/w_2 \notin \mathbb{R}$ and $f(z + w_1) = f(z) = f(z + w_2)$ for all $z \in \mathbb{C}$).

c) Let $f(z)$ be a doubly periodic function with periods $w_1, w_2$. Let $a \in \mathbb{C}$ be such that $f(z)$ has no poles on the boundary of the parallelogram

$$\Gamma$$

whose vertices are at $a, a + w_1, a + w_2, a + w_1 + w_2$.

Show that $f(z)$ has finitely many poles $\{z_1, ..., z_N\}$ in the interior of $\Gamma$ and that

$$\sum_{i=1}^{N} \operatorname{Res}(f; z_i) = 0.$$