M.E.T.U

Department of Mathematics

Preliminary Exam - Sep. 2011

COMPLEX ANALYSIS

Duration : 180 min.

Each question is 25 pt.

1. For $R > 0$, let Γ_R be the counterclockwise oriented closed curve obtained by joining the following paths, in the given order :

   $\Gamma_{1,R} : [0, R] \to \mathbb{C}$ defined by $\Gamma_{1,R}(t) = t$
   $\Gamma_{2,R} : [0, \pi/2] \to \mathbb{C}$ defined by $\Gamma_{2,R}(t) = Re^{ti}$
   $\Gamma_{3,R} : [0, R] \to \mathbb{C}$ defined by $\Gamma_{3,R}(t) = (R - t)i$.

   Consider the analytic function $f : \mathbb{C} - \{0\} \to \mathbb{C}$ defined by

   $$f(z) = \frac{e^{-z} - e^{iz}}{z}$$

   for each $z \neq 0$.

   (A) Show that $z = 0$ is a removable singularity of $f(z)$.

   (B) Prove that

   $$\int_{\Gamma_R} f(z)dz = 0 \ .$$

   (C) Prove that

   $$\lim_{R\to\infty} \int_{\Gamma_{2,R}} f(z)dz = 0 \ .$$

   (D) Prove that

   $$\int_0^\infty \frac{e^{-x} - \cos x}{x}\, dx = 0 \ .$$

2. Consider

$$\Omega = \{z \in \mathbb{C} \mid \operatorname{Im}(z) > 0\}$$

and

$$\Delta = \{z \in \mathbb{C} \mid |\, z\, | < 1\} \ .$$

(A) Given any $b \in \Omega$ prove that the map $\Psi = \Psi_b : \mathbb{C} - \{\bar{b}\} \to \mathbb{C}$ defined for each $z \neq \bar{b}$ by

$$\Psi(z) = \frac{z-b}{z-\bar{b}}$$

maps Ω onto Δ bijectively.

(B) If $f : \Omega \to \mathbb{C}$ is analytic and satisfies $f(\Omega) \subseteq \Omega$, prove that

$$\frac{|f(z) - f(a)|}{|f(z) - \overline{f(a)}|} \leq \frac{|z-a|}{|z-\bar{a}|}$$

for any $z, a \in \Omega, z \neq a$. [1]

(C) Deduce that

$$|f'(z)| \leq |\frac{\operatorname{Im}(f(z))}{\operatorname{Im}(z)}|$$

for any $z \in \Omega$.

[1] Consider $g = \Psi_{f(a)} \circ f \circ \Psi_a^{-1}$.

3. Let $\Omega \subset \mathbb{R}^2 \cong \mathbb{C}$ be an open connected region and let

$$f = (u, v) : \Omega \to \mathbb{C}$$

be a **non-constant differentiable** function. Consider the set

$$Z_{df} = \{z = (x, y) \in \Omega : det(df(x, y)) = 0\}.$$

a) Give an example $f(x, y)$ for which Z_{df} is not a discrete subset of Ω.

b) Suppose that $f(z)$ is an analytic function.

- Show that Z_{df} is discrete.
- Suppose that $z_0 \in Z_{df}$ and that $f(z_0) = 0$. Show that there exists an integer $n > 1$ such that the function $g(z) = 1/f(z)$ maps a neighborhood of z_0 analytically onto a neighborhood of ∞ in an n-to-one manner.
- Show that if $\text{Res}(g; z_0) = 0$ then $g(z)$ is the derivative of a function meromorphic around z_0.

4. a) Show that if $f(z)$ is a non-constant entire periodic function, then $f(z)$ has an essential singularity at ∞.

b) Does there exist a non-constant entire doubly periodic function ? Explain.

(Recall that a meromorphic function $f(z)$ is said to be doubly periodic with periods w_1, w_2 if $w_1/w_2 \notin \mathbb{R}$ and $f(z + w_1) = f(z) = f(z + w_2)$ for all $z \in \mathbb{C}$).

c) Let $f(z)$ be a doubly periodic function with periods w_1, w_2. Let $a \in \mathbb{C}$ be such that $f(z)$ has no poles on the boundary of the parellogram

$$\Gamma \text{ whose vertices are at } a,\ a + w_1,\ a + w_2,\ a + w_1 + w_2.$$

Show that $f(z)$ has finitely many poles $\{z_1, ..., z_N\}$ in the interor of Γ and that

$$\sum_{1}^{N} \text{Res}(f; z_i) = 0.$$