PRELIMINARY EXAM - Sep.2012
Complex Analysis

Duration : 3 hr.

Q.1	Q.2	Q.3	Q.4	Total

1. (25 = 20+5 pt.)

   a) Let
   $$f : \overline{D}(0;1) \to \mathbb{C} \quad \text{and} \quad g : \overline{D}(0;1) \to \mathbb{C}$$
   be two continuous functions which are analytic on $D(0;1)$. Show that if $f = g$ on the unit circle $|z| = 1$, then $f = g$.

   b) Give an example of a pair of smooth functions f, g on $\overline{D}(0;1)$ such that $f \neq g$ on $D(0;1)$ but $f(x,y) = g(x,y)$ on the circle $x^2 + y^2 = 1$.

2. (25 = 7+10+8 pt.)

   Let $\Omega \subset \mathbb{C}$ be an open region and let $f(z)$ be a function which is analytic on Ω except for a set of isolated singularities.

   a) Show that $\text{Residue}(f'(z); z_0) = 0$ for all $z_0 \in \Omega$.

   b) Show that if $f(z)$ is meromorphic on Ω, then the function
   $$g(z) = e^{f(z)}$$
   has no poles in Ω.
   (Hint : For $z_0 \in \Omega$, apply the principle of argument to $g(z)$ in a suitable neighbourhood of z_0).

   c) Construct an analytic function $h : \mathbb{C} - \{n\pi : n \in \mathbb{Z}\} \to \mathbb{C}$ which has an essential singularity at each point $z_n = n\pi, n \in \mathbb{Z}$.

3. (25 = (7 + 3) + 7 + 8 pt.)
$\mathsf{n}(\gamma, \mathsf{a})$ denotes the *index* at $a \in \mathbb{C}$ of the closed curve $\gamma : [0, 2\pi] \to \mathbb{C} - \{a\}$.

(A) If $b \neq 0$, prove that

$$\mathsf{n}(\gamma^n, b^n) = \mathsf{n}(\gamma, b).$$

Prove also that

$$\mathsf{n}(\gamma^n, 0) = n\,\mathsf{n}(\gamma, 0)$$

for $n \in \mathbb{Z}$ with $n \geq 1$.

(B) If $a \in \mathbb{C} - \{0\}$ is an isolated singularity of analytic $f(z)$, prove that any b with $a = b^n$ is an isolated singularity of $g(z) = f(z^n)$.

(C) Show that $\frac{\mathrm{Res}_{z=a}(f(z))}{a} = n \frac{\mathrm{Res}_{z=b}(g(z))}{b}$

4. (25 = 5 + 8 + 7 + 5 pt.)
Consider an open $\Omega \subseteq \mathbb{C}$ and $a \in \Omega$. Let $\Delta = \{z \in \mathbb{C} \mid |z| < 1\}$.

(A) If $h : \Omega \to \mathbb{C}$ is a continuous function which is analytic on $\Omega - \{a\}$, prove that h is analytic on Ω.

(B) Let $f : \mathbb{C} - \Delta \to \Delta$ be an analytic function with $\lim_{z \to \infty} f(z) = 0$.
Prove that

$$|f(z)| \leq \frac{1}{|z|}$$

for all $z \in \mathbb{C} - \Delta$.

(C) Prove that $\lim_{z \to \infty} \Big(z f(z) \Big)$ exists and $\Big| \lim_{z\to\infty} \Big(z f(z) \Big) \Big| \leq 1.$

(D) Prove that in order for any one of the inequalities in (B) and (C) to become an equality, it is necessary and sufficient that f is a constant.