Complex Analysis

PRELIMINARY EXAMINATION

Monday, 15th September 2014. Four questions, three hours.

1

[12 + 7 + 6]

(A) Prove that the real and imaginary parts of $f : \mathbb{C} \to \mathbb{C}$ defined by

$$f(z) = \begin{cases} \exp\left(-\dfrac{1}{z^4}\right) & \text{for} \quad z \neq 0 \\ 0 & \text{for} \quad z = 0 \end{cases}$$

have partial derivatives that satisfy the Cauchy-Riemann equations at any $z \in \mathbb{C}$.

(B) Is $f(z)$ differentiable as a function of z at $z = 0$?

(C) Give conditions under which the Cauchy-Riemann equations are sufficient for differentiability with respect to z. Indicate why they are not applicable in this case.

2

[(7 + 7) + 11]

(A) Given an entire function f evaluate

$$\int_{\Gamma_R} \frac{f(z)}{(z-a)(z-b)}\, dz$$

for any $a, b \in \mathbb{C}$ where $R > |a|, |b|$ and the contour Γ_R is the counterclockwise traversed circle of center $0 \in \mathbb{C}$ and radius R. Use this relation to prove the Liouville theorem to the effect that a bounded entire function reduces to a constant.

(B) Let h be an entire function that h has simple zeros, only. If g is an entire function which satisfies

$$|g(z)| \leq |h(z)|$$

for all $z \in \mathbb{C}$, prove that $g(z) = ch(z)$ for some constant $c \in \mathbb{C}$ with $|c| \leq 1$.

3

[15 + 10]

(A) Prove that the polynomial

$$p(z) = a_0 + a_1 z + a_2 z^2 + \cdots + a_n z^n$$

where $a_0, a_1, a_2, \cdots, a_n \in \mathbb{R}$ with

$$a_0 \geq a_1 \geq a_2 \geq \cdots \geq a_n > 0$$

has no roots in the open disc

$$\Delta = \left\{ z \in \mathbb{C} \;\middle|\; |z| < 1 \right\} \quad .$$

(*Hint* : Consider the polynomials $(z-1)p(z)$ and $a_n z^{n+1} - a_0$ on the boundary of Δ .)

(B) Prove that the polynomial

$$q(z) = b_0 + b_1 z + b_2 z^2 + \cdots + b_n z^n$$

where $b_0, b_1, b_2, \cdots, b_n \in \mathbb{R}$ with

$$b_n \geq \cdots \geq b_1 \geq b_0 > 0$$

has exactly n roots (counted with multiplicities) in the closure of Δ .

4

[8 + 17]

Use your own choice of branches of the squareroot and the logarithmic function to prove by calculus of residues, or otherwise, that

$$\int_0^\infty \frac{\sqrt{x}\log(x)}{x^2+1}\, dx = \frac{\pi^2}{2\sqrt{2}} \quad .$$