Preliminary Exam - September 2015
COMPLEX ANALYSIS

Each question is 25 pt.

1. Consider the entire function $f(z) = z^2 + Bz$.

   a) Determine B if $|f(B)| = |B|$ and at $z = 0$ the given map defines a rotation through $\theta = \pi/4$.

   b) Determine the zeros, poles and all multiple points of the holomorphic map

   $$f : \hat{\mathbb{C}} \to \hat{\mathbb{C}}.$$

   c) Let $g(z) = f(z)^{-n}$, $n \geq 1$. Determine

   $$\int_{|z|=R} \frac{g'(z)}{g(z)} dz$$

   for all permissible values of R.

2. Consider the function $f(z) = e^{1/z(z-1)}$.

   a) Show that f is holomorphic in $\mathbb{C} - \{0,\ 1\}$ with essential singularities at $z = 0$ and $z = 1$.

   b) Compute

   $$\int_{z=R} f(z) dz$$

   for

   (i) $0 < R < 1$, (ii) $R > 1$.

   c) True or false ? Why ?

   For any pair of positive real numbers $\{r,\ r'\}$, $f(D^*(0; r)) \cap f(D^*(1; r')) \neq \emptyset$.

3. Let $\sum_0^\infty a_n z^n$ be a power series which converges in $\mathbb{C}$ to a nowhere vanishing function $f(z)$.

   a) Show that for any given $R > 0$, there exists a positive integer N such that for all $m \geq N$, the polynomial $f_m(z) = \sum_0^m a_n z^n$ has no zeros in $D(0; R)$.

   b) True or false ? Explain.

   (i) $log(f(z))$ can be defined as an entire function.

   (ii) f extends to a meromorphic function on the extended complex plane if and only if it is constant.

4. a) Show that if $f : \overline{D}(0; R) \to \mathbb{C}$ is holomorphic and $|f(z)| < R$ on $|z| = R$, then there exists a unique point $a \in D(0; R)$ such that $f(a) = a$.

   b) Show that for $a,\ b \in \Omega$ (Ω a simply connected region), there exists a holomorphic automorphism

$$\Phi : \Omega \to \Omega$$

   such that $\Phi(a) = b$. Is Φ unique ?