Complex Analysis Preliminary Exam
Fall 2020

1. (a) (10pts) Let $a \in \mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$. Show that $\phi(z) = \frac{z-a}{1-\bar{a}z}$ is a conformal self -map (holomorphic, one-to-one and onto) of $\mathbb{D}$.

   (b) (15 pts) Let $f : \mathbb{D} \to \mathbb{D}$ be a holomorphic (analytic) function with zeros at $i/3$ and $1/4$. Show that $|f(0)| \leq \frac{1}{12}$.

2. (25pts) Show that, if f is analytic on the unit disk $\mathbb{D}$, $f^{(n)}(0) \in \mathbb{R}$ and $f^{(n)}(0) \geq \frac{n!}{n^2}$ for all $n \geq 1$, then f does not extend analytically near $z = 1$.

3. (a) (10pts) Find all conformal self-maps of the complex plane.

   (b) (15pts) Find a conformal map from $\{z \in \mathbb{C} : Re\ z - 2 < Im\ z < Re\ z - 1\}$ onto the unit disk $\mathbb{D}$.

4. (a) (15pts) Find the number of zeros of $f(z) = z^3 + 3z^2 + 21$ on the disk $|z - i| < 1$.

   (b) (10pts) Calculate

$$\int_{|z-i|=1} \frac{dz}{(z^2+1)(z^3+3z^2+21)}.$$