GRADUATE PRELIMINARY EXAMINATION
ANALYSIS 2 (Complex Analysis)

February 17th, 2005

1. Let G be the group of analytic automorphisms $g : D(0:1) \to D(0:1)$ of the open unit disc D(0:1) onto itself.
   (a) For any two elements z_1, z_2 in D(0:1), explicitly construct $g \in G$ such that $g(z_1) = z_2$.
   (b) Characterize the elements of $G_0 = \{g \in G : g(0) = 0\}$.
   (c) Determine all holomorphic functions $f : D(0:1) \to \mathbb{C}$ which are G-invariant, i.e. $f(g(z)) = f(z) \forall g \in G \ , z \in D(0:1)$.
   (d) Determine all holomorphic functions $f : D(0:1) \to \mathbb{C}$ which are G_0-invariant.

2. Let f be an entire function which satisfies
$$f(z) + f(z+1) = f(2z) \ \ \forall z \in \mathbb{C}.$$
   (a) Using induction on n show that
$$f(2^n z) = \sum_{k=0}^{2^n-1} f(z + \frac{k}{2^{n-1}}) \ \forall n \in \mathbb{Z}, n \geq 1$$
   (b) Let D(0,r) denote the open unit disc with center at $0 \in \mathbb{C}$ and radius $r > 0$. Using the Cauchy Integral Formula over the counterclockwise oriented circle of radius 2^n centered at 0 or otherwise, show that for any $a \in D(0,1)$ and $n \in \mathbb{Z}, n \geq 1$
$$|f''(a)| \leq \frac{M}{2^{n-4}}$$
   where $M = \sup_{z \in D(0:3)} |f(z)|$.
   (c) Prove that $f(z) = Az + B$ for some $A, B \in \mathbb{C}$ with A+B=0.

3. Consider the series
$$f(z) = \sum_{n=0}^{\infty} z^{n!}.$$
   (a) Show that f(z) defines an analytic function in the open unit disc D(0,1).
   (b) Verify that for all $k \geq 1$, and for all k-th roots of unity w, (i.e. $w^k = 1$), $f(w) = \infty$ holds.
   (c) Show that in any arc on the unit circle $|z| = 1$, there are N-th roots of unity for infinitely many N. (*Hint: Use the map* $[0,1] \to \{z : |z| = 1\}, t \longmapsto e^{2\pi i t}$ *to work in* [0,1]).
   (d) Using the results of b) and c) show that f(z) cannot be continued analytically to any domain Ω which properly contains D(0,1).

4. Let Ω be a convex bounded domain and γ a closed smooth curve in Ω. Suppose that f and g are analytic functions on $\overline{\Omega}$, f zero free on γ.
   (a) Compute the residue of $\frac{g.f'}{f}$ at a zero of f in Ω.
   (b) Compute $\frac{1}{2\pi i} \oint_\gamma \frac{g(z)f'(z)}{f(z)} dz$.