METU - Mathematics Department
Graduate Preliminary Exam-Spring 2008

Complex Analysis

NOTATION :

$D = \{z \in \mathbb{C} : |z| < 1\}$.
Unless otherwise stated, Ω denotes an open connected set in $\mathbb{C}$.
For a region Ω, $\mathrm{Aut}(\Omega)$ denotes the group of holomorphic automorphisms of Ω.

1. A) Let $f : \Omega \to \mathbb{C}$ be a function with

$$f(z) = u(x, y) + iv(x, y)$$

for any $z = x + iy \in \Omega$. Prove that for each $\alpha + i\beta \in \Omega$ at which $f(z)$ is differentiable as a function of z, the functions u, v have partial derivatives at (α, β) which satisfy the Cauchy-Riemann equations.

B) Prove that $f : \mathbb{C} \to \mathbb{C}$, defined by

$$f(z) = x^3 + i(1 - y)^3$$

is differentiable only at $z = i$. Evaluate $f'(i)$.

C) Prove that the real and imaginary parts of $f : \mathbb{C} \to \mathbb{C}$ defined by

$$f(z) = \begin{cases} \dfrac{x^3(1+i) - y^3(1-i)}{x^2 + y^2} & \text{for } z \neq 0 \\ 0 & \text{for } z = 0 \end{cases}$$

satisfy the Cauchy-Riemann equations at $(0, 0) \in \mathbb{R}^2 \cong \mathbb{C}$ but f is not differentiable at $0 \in \mathbb{C}$.

2. A) Let $f : \Omega \to \mathbb{C}$ be an analytic function. Given $a \in \Omega$, prove that for $m \in \mathbb{N}$

$$\mathrm{Res}_{z=a}\left(\frac{f(z)}{(z-a)^{m+1}} \right) = \frac{1}{m!} \frac{d^m f(z)}{dz^m} \Big|_{z=a}.$$

B) Compute

$$\text{Res}_{z=i}\left(\frac{e^{iz}}{(z^2+1)^2}\right).$$

C) Prove that

$$\int_{-\infty}^{\infty}\frac{\cos x}{(x^2+1)^2}dx = \frac{\pi}{e} \quad .$$

3. Let $z_1, z_2 \in D$ be any two distinct points and let $\Omega = D - \{z_1, z_2\}$.

   a) Show that any analytic map $f : \Omega \to D$ extends to an analytic map $D \to D$.

   b) Show that in part (a) if f is an isomorphism, then so is the extension.

   c) Find a relation between z_1 and z_2 which is necessary and sufficient in order to have an isomorphism $\Omega \to D - \{0, 1/2\}$.

4. True or false ? Prove the statement or give a counter example.

   a) If $f(z)$ is a non-constant entire function such that $|f(z)|$ is bounded on $\mathbb{R}$, then $f(z)$ has an essential singularity at ∞.

   b) If $f(z)$ is meromorphic in $\mathbb{C} \cup \{\infty\}$, then it is a rational function (ie. the ratio of two polynomials).

   c) Let $p(z)$ be a polynomial such that for all sufficiently large R we have

$$\int_{|z|=R}\frac{p'(z)}{p(z)}dz = 2\pi i N, \text{ for some } N \geq 1.$$

   Then $p(z)$ defines a surjective holomorphic mapping $\mathbb{C} \cup \{\infty\} \to \mathbb{C} \cup \{\infty\}$ of degree N.