PRELIMINARY EXAM - Feb.2012
Complex Analysis

Duration: 3 hr.

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1. \((5+7+13=25\ \text{pt.})\) Let \(n \in \mathbb{N}\) with \(n \geq 2\) and \(\omega = e^{\pi i/n}\).
   
   (A) Show that \(\omega^{\frac{n(n-1)}{2}} = i^{n-1}\).
   
   (B) Show that \(\frac{x^n - 1}{x - 1} = \prod_{k=1}^{n-1} (x - \omega^{2k})\) for every \(x \neq 1\).
   
   (C) Prove that 
   \[
   \prod_{k=1}^{n-1} \cos \left( \frac{k\pi}{n} \right) = \begin{cases} 
   0 & \text{if } n \text{ is even} \\
   \frac{(-1)^{\frac{n-1}{2}}}{2^{n-1}} & \text{if } n \text{ is odd}
   \end{cases}
   \]

2. \((4+8+13=25\ \text{pt.})\)
   
   (A) Prove that \(|e^z| = e^{\text{Re}(z)}\)
   
   (B) Let \(f\) be an entire function such that \(|f(z)| \leq e^{\text{Re}(z)}\). Show that there exists a constant \(a \in \mathbb{C}\) such that 
   \(f(z) = ae^z\).
   
   (C) Let \(g\) be an entire function such that \(g(z+1) = -g(z)\), \(g(0) = 0\) and 
   \(|g(z)| \leq e^{\text{Im}(z)}\).
   
   Show that there exists a constant \(b \in \mathbb{C}\) such that 
   \(g(z) = b \sin (\pi z)\).

3. \((8+10+7=25\ \text{pt.})\) Let \(\Omega \subset \mathbb{C}\) be a domain and \(f(z)\) be a meromorphic function in \(\Omega\) with a non-empty set \(W\) of poles. Choose an arbitrary point \(z_0 \in \Omega - W\).
   
   a) Show that \(W\) is a discrete subset of \(\Omega\).
   
   Give an example where \(\Omega\) is bounded and \(W\) is an infinite set.
   
   b) Show that if the residue of \(f(z)\) at each pole vanishes, then
• for \( z \in \Omega - W \) the integral

\[
F(z) = \int_{z_0}^{z} f(u)du
\]

is independent of the path \( \Gamma \subset \Omega - W \) connecting \( z_0 \) and \( z \), and
• \( F(z) \) defines an analytic function in \( \Omega - W \).

c) True or false? Explain.

\( F(z) \) is meromorphic in \( \Omega \) with \( W \) as the set of poles.

4. (10+8+7 = 25 pt.) Let \( g(z) \) be a non-constant entire periodic function, \( f(z) \) be a meromorphic function in \( \mathbb{C} \).

a) Let \( z_0 \) be a pole of \( f(z) \). Show that the function \( g \circ f \) has an essential singularity at \( z_0 \) (that is, \( \lim_{z \to z_0} (g \circ f(z)) \) does not exist).

b) For \( g(z) = e^z \), prove the result in (a) by using the argument principle.

c) Show that if \( f(z) \) has at least two poles then \( f \circ g \) has infinitely many poles.