Preliminary Exam - February, 2015
COMPLEX ANALYSIS

Each question is 25 pt.

1. a) Show that the function

$$f : \mathbb{R}^2 \to \mathbb{R}^2,\ f(x,y) = (\sin(2x+y), \cos(2x+y))$$

is differentiable everywhere.

Determine all points, if any, where f has a local differentiable inverse.

b) Show that as a complex valued function of the complex variable $z = x + iy$, $f(z)$ is nowhere analytic by

(i) checking the Cauchy-Riemann conditions,

(ii) using the topological mapping defined by f.

c) Can you find an entire function $g(z)$ such that $\mathrm{Re}(g(z)) = \sin(2x+y)$? Explain.

2. Let $f,\ g$ be two entire functions which have simple zeros precisely at

$$z_n = \frac{(2n+1)}{2}\pi,\ n \in \mathbb{Z}$$

and no other zeros.

a) Show that the function $\dfrac{f}{g}(z)$ is entire.

b) Show that $\dfrac{f}{g}(z) = e^{h(z)}$ for some entire function $h(z)$.

c) For $N \geq 1$, let $a_N = \displaystyle\int_\gamma \frac{f'(z)}{f(z)} dz$ where γ is the circle $|z| = 1 + N\pi$.

True or false ? Explain.

The series $\sum_1^\infty \dfrac{1}{a_N^{1+r}}$ converges for all $r > 0$.

3. a) Show that if $\Omega \subset \mathbb{C}$ is a domain such that $\mathrm{Aut}(\Omega)$ is a finite group, then Ω is not simply connected.

   b) True or false ? Give an example or disprove the statement.

   There exists an analytic function $f : \mathbb{C} - 0 \to \mathbb{C}$ such that $f(\mathbb{C} - 0) = \mathbb{C}$ and f has an essential singularity at $z = 0$.

   c) Show that the family of functions $f_n(z) = z^n, n \geq 1$ is normal in $D(0; 1)$, but not in any region which contains a point on the unit circle.

4. Let $S = \{z_n \in \mathbb{C} : n \geq 1\}$ be a discrete set and let

$$f : \mathbb{C} - S \to \mathbb{C}$$

   be a holomorphic function.

   a) Suppose that $\mathrm{Res}(f; z_1) = 0$. Show that there exist $R > 0$ and a holomorphic function $g(z)$ in $D^*(z_1; R)$ such that $f(z) = \dfrac{dg}{dz}(z)$.

   b) Show that $f(z) = \dfrac{dh}{dz}(z)$ for some analytic function $h(z)$ in $\mathbb{C} - S$ if and only if $\mathrm{Res}(f; z_n) = 0$, for all $n \geq 1$.