TMS - Complex Analysis
Feb. 2018

Each question is 25 pts.

Notation :
$D = \{z : |z| < 1\}$, $D^* = \{z : 0 < |z| < 1\}$, $\mathbb{C}^* = \mathbb{C} - \{0\}$, $\mathbb{P}^1 = \mathbb{C} \cup \{\infty\}$.
$\mathcal{H}, \overline{\mathcal{H}}$ denote respectively, the upper half plane and its closure in $\mathbb{R}^2$.
$\dfrac{\partial(u,v)}{\partial(x,y)}(p)$ is the jacobian at p of the function $(x,y) \mapsto (u(x,y), v(x,y))$.

1. Let $f(z) = u(x,y) + iv(x,y)$ be a non-constant holomorphic function in a connected region $\Omega \subset \mathbb{C}$.

   a) Show that the set
   $$S = \{p \in \Omega : \frac{\partial(u,v)}{\partial(x,y)}(p) = 0\}$$
   is discrete in Ω.

   b) Show by an example that even if Ω is bounded, S need not be a finite set.

   c) If $z_0 \in S$ and $\dfrac{d^2 f}{dz^2}(z_0) \neq 0$, then in a disc around z_0 we can write
   $$f(z) - f(z_0) = \omega(z)^2$$
   for a suitable one-to-one holomorphic function ω of z.

2. a) By using complex analysis, prove that there does not exist a pair of non-constant continuous functions $f_j : \overline{\mathcal{H}} \to \mathbb{R}$, $j = 1, 2$ which satisfy the following conditions:

   In $\mathcal{H}$, f_1, f_2 are harmonic conjugate functions and $|f_1|, |f_2|$ are bounded.

   At least one of the functions f_1, f_2 vanishes identically on the boundary of $\overline{\mathcal{H}}$.

   b) Give an example of a pair of harmonic conjugate functions in $\mathcal{H}$ which satisfy all the conditions in (a) except the boundedness of $|f_1|, |f_2|$.

   c) Give an example of a pair of harmonic conjugate functions in $\mathcal{H}$ which satisfy all the conditions in (a) except the vanishing of f_1 or f_2 on the boundary.

3. Let $\Omega \neq \mathbb{C}$ be a simply connected region and $z_0 \in \Omega$ be an arbitrary point.

   a) True or false ? Explain.

   For each positive real number r there exists a unique analytic isomorphism $f_r : \Omega \to D$ such that $f_r(z_0) = 0, f'(z_0) = r$.

b) Fix an isomorphism $f : \Omega \to D,\ f(z_0) = 0$. Show that

$$G = \{\phi \in Aut_{hol}(\Omega) : \phi(z_0) = z_0\}$$

consists of maps

$$\phi : \Omega \to \Omega,\ \phi(z) = f^{-1}(e^{i\theta} f(z)),\ \theta \in [0, 2\pi).$$

c) Show that for any $z_1 \in \Omega,\ z_1 \neq z_0$ we have $\phi \in Aut_{hol}(\Omega)$ such that $\phi(z_1) = z_0$. Is ϕ unique ?

4. f is a meromorphic function in $\mathbb{C}$ with poles at $z_j,\ j = 1, .., n$ and is holomorphic elsewhere. Suppose that for any circle $C_j : |z - z_j| = r$ which does not pass through any pole of f and encloses only z_j and no zeros of $f(z)$, we have

$$\frac{1}{2\pi i} \int_{C_j} \frac{f'(z)}{f(z)} dz = -j.$$

a) Suppose that f extends to a meromorphic function $\overline{f} : \mathbb{P}^1 \to \mathbb{P}^1$.

- Determine the minimum possible value of $d(\overline{f})$ (the degree of $\overline{f}$).
- True or false ? Why ? We have $d(1/\overline{f}) = d(\overline{f})$.

b) Determine the general form of f

- if $g(z) = \prod_1^n (z - z_j)^j f(z)$ is holomorphic at ∞.
- if $g(z) = \prod_1^n (z - z_j)^j f(z)$ has no zeros in $\mathbb{C}$.