1. (a) (5pts) State Rouche’s Theorem.
(b) (15 pts) Let $f$ be a holomorphic function in $U \subset \mathbb{C}$. If $f'(z_0) = 0$ for some $z_0 \in U$ then one can find $r > 0$, $\rho > 0$ such that, $f'(z) \neq 0$, $\forall z \in \{0 < |z - z_0| < r\} \subset U$, and $|f(z) - f(z_0)| > \rho$, $\forall z : |z - z_0| = r$. (Explain why?) Then show that for any $w$ such that $|w - f(z_0)| < \rho$, $f(z) - w$ has at least 2 distinct zeroes in the disk $\{|z - z_0| < r\}$.
(c) (5 pts) Note that in part (b) you showed that if $f$ is one-to-one in $U$ then $f'$ has no zeroes. Is the converse true? That is, if $f'$ never vanishes in $U$, then is it true that $f$ must be one-to-one in $U$?

2. (25pts) Show that, if $f$ is analytic on the unit disk $D = \{z \in \mathbb{C} : |z| < 1\}$ and continuous on $\overline{D}$ with $|f(z)| = 1$ on $\partial D$, then $f$ is a rational function, that is $f = P/Q$ where $P, Q$ are polynomials.

3. (a) (15pts) Find a conformal map from the strip $\{z \in \mathbb{C} : 0 < Re z < 1\}$ onto the unit disk $\mathbb{D}$.
(b) (10pts) Show that $\mathbb{C}$ and the upper half-plane $\mathbb{H} = \{z \in \mathbb{C} : Im z > 0\}$ are not biholomorphic. (Two subsets are called biholomorphic if there is a conformal map between them)

4. (a) (10pts) Show that $f(z) = \sum_{n=0}^{\infty} z^{2^n}$ is analytic on the open unit disk.
(b) (15pts) Show that $f$ can not be extended analytically to any open set which is larger than the unit disk. (Hint: First show that $f(z) = z + f(z^2)$ and hence (why?) $f(r) \to \infty$ as $R \ni r \to 1^-$. Then by considering the roots of unities, explain why this implies that $f$ can not be extended through any point on the boundary of the unit disk.)