Complex Analysis Preliminary Exam
Spring 2021

1. (a) (10pts) Let D be a domain in $\mathbb{C}$ and $z_0 \in D$. Find all analytic functions on D satisfying $f^{(n)}(z_0) = 0$ for all $n \geq 1$.

   (b) (15 pts) Show that if an analytic function f has a zero of order N at a point z_0, then $f(z) = g(z)^N$ for some function g analytic near z_0.

2. (25pts) Show that, if f is analytic on the unit disk $\mathbb{D}$, $f^{(n)}(0) \in \mathbb{R}$ and $f^{(n)}(0) \geq n^{-2021}$ for all $n \geq 1$, then f does not extend analytically near $z = 1$.

3. (25 points) Let $\phi(z)$ be a conformal map of a simply connected domain D onto the open unit disk $\mathbb{D}$ satisfying $\phi(z_0) = 0$ for some $z_0 \in D$. Show that if $f : D \to \mathbb{D}$ is analytic then $|f'(z_0)| \leq |\phi'(z_0)|$.

4. (a) (10pts) Find a conformal map from $\{z \in \mathbb{C} : Imz > Rez - 1\}$ onto the unit disk $\mathbb{D}$.

   (b) (15pts) For any $k \in \mathbb{Z}$, calculate

$$I_k = \int_{|z|=1} z^k \sin(1/z) dz.$$