METU-MATHEMATICS DEPARTMENT
Graduate Preliminary Examinations

Geometry

Duration: 3 hours

September 24, 2004

1. Consider $(0,2)$-tensor field T and a $(1,1)$-tensor field S on $\mathbb{R}^2$, with the components $T_{i,j} = S^i_j = i - j + 2,\ i,j = 1,2$, where $\mathbb{R}^2$ is considered as a manifold with usual coordinates (i.e. with coordinates with respect to the standard basis e_1, e_2)

(**a**) Determine the components $T_{\alpha\beta}$ S^α_β of T and S when the coordinates in $\mathbb{R}^2$ are considered with respect to the basis $f_1 = e_1 + e_2$ and $f_2 = 2e_1 + e_2$

(**b**) Determine the components of $\operatorname{Alt} T$ and $\operatorname{Sym} T$ with respect to the basis e_1, e_2).

2. For each point $p = [u,v,w]$ on $\mathbb{R}P^2$ define curves γ_p and σ_p by

$$\begin{aligned} \gamma_p(t) &= [u, e^{-t}v, e^{-t}w] \\ \sigma_p(t) &= [u\cos t - v\sin t, u\sin t + v\cos t, w] \end{aligned}$$

for $t \in \mathbb{R}$. Consider the vector fields $A, B \in \mathfrak{X}(\mathbb{R}P^2)$ which assigns the values $\gamma_p'(0)$ and $\sigma_p'(0)$ respectively to each point $p \in \mathbb{R}P^2$

(**a**) Introduce a chart of your own choice on $\mathbb{R}P^2$ and find local expressions for A, B on this chart.

(**b**) Find local expressions for the Lie bracket $[A, B]$ on the same chart.

(**c**) For each point $p = [u,v,w]$ on $\mathbb{R}P^2$ find a curve $\theta_p : \mathbb{R} \to \mathbb{R}P^2$ such that $\theta_p(0) = p$ and $[A,B]$ takes the value $\theta_p'(0)$ at the point $p \in \mathbb{R}P^2$.

3. Consider the two dimensional sphere

$$\mathbf{S}^2 = \{(u,v,w) \in \mathbb{R}^3 \mid u^2 + v^2 + w^2 = 1\} \subseteq \mathbb{R}^3$$

with its usual smooth structure and the smooth maps $f, g : \mathbf{S}^2 \to \mathbb{R}$ defined by

$$\begin{aligned} f((u, v, w)) &= w \\ g((u, v, w)) &= u \end{aligned}$$

(**a**) Evaluate the integral

$$\int_M df \wedge dg$$

where M is the manifold with boundary defined by

$$M = \{(u, v, w) \in \mathbf{S}^2 \mid v \geq 0\}$$

<u>without</u> employing Stokes' theorem.

(**b**) Use Stokes' theorem to evaluate the same integral.

4. Let M be a compact manifold and let $f : M \to N$ be a submersion where N is an arbitrary manifold with $\dim M = \dim N$. Define a function $\varphi : N \to \mathbb{R} \cup \{\infty\}$ by

$$\varphi(y) = \text{number of points in } f^{-1}(y)$$

(**a**) Prove that $\varphi(y)$ is finite for each $y \in N$.

(**b**) Prove that $\varphi : N \to \mathbb{R}$ is a locally constant function.