1. **a)** Show that a one-to-one immersion of a **compact** manifold is an imbedding.

**b)** Explain, in full details, why the map \( \phi : (-\pi, \pi) \to \mathbb{R}^2, \phi(s) = (\sin(2s), \sin(s)) \) shows that the conclusion in part (a) is false if \( X \) is not compact.

2. Let \( SL_n(\mathbb{R}) \) denote the \( n \times n \) real matrices with determinant 1.

   **a)** Show that \( SL_n(\mathbb{R}) \) is a submanifold of the \( n \times n \) matrices \( M_n(\mathbb{R}) \).

   **b)** Show that the tangent space to \( SL_n(\mathbb{R}) \) at the identity matrix \( I \) is \( T_I SL_n(\mathbb{R}) = \{ A \in M_n(\mathbb{R}) : \text{trace}(A) = 0 \} \).

3. **a)** What is meant by an orientation on a manifold?

   **b)** Show that \( S^n = \{ \mathbf{x} \in \mathbb{R}^{n+1} : |\mathbf{x}| = 1 \} \) is an oriented manifold, by defining an orientation on it.

   **c)** Show that the antipodal map \( S^n \to S^n, \mathbf{x} \mapsto -\mathbf{x} \) is orientation preserving if and only if \( n \) is odd.

   **d)** Using (c), or otherwise show that \( \mathbb{R}P^n \) is orientable if and only if \( n \) is odd.

4. **a)** Show that \( X = \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 = 1 \} \) is a closed submanifold of \( \mathbb{R}^3 \).

   **b)** Verify that the restriction \( \omega|_X \) of \( \omega = \frac{xdy - ydx}{x^2 + y^2} \) is a closed 1-form on \( X \).

   **c)** Calculate \( \int_S \omega|_X \), where \( S \) is the circle \( \{(x, y, 3) : x^2 + y^2 = 1 \} \subset X \).

   Is \( \omega|_X \) an exact form? Why?

   **d)** Consider the mapping \( \Psi : \mathbb{R}^2 \to X, \Psi((s, t)) = (\cos(s), \sin(s), t) \). Show that \( \Psi \) is a differentiable map and that the form \( \Psi^*(\omega|_X) \) is exact.